A NUMERICAL TREATMENT OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS OF ELLIPTIC TYPE USING THE BOUNDARY ELEMENT METHOD

by

K. ARUMUGAM

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MARCH, 1997

A NUMERICAL TREATMENT OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS OF ELLIPTIC TYPE USING THE BOUNDARY ELEMENT METHOD

A Thesis Submitted

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

by

K. ARUMUGAM

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

March, 1997

Dedicated to
My Parents

CERTIFICATE

It is certified that the work contained in this thesis entitled *A Numerical Treatment of Singularly Perturbed Boundary Value Problems of Elliptic Type Using the Boundary Element Method* by *K. ARUMUGAM* has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

26/3/9

Professor M. K. KADALBAJOO
Department of Mathematics
Indian Institute of Technology
Kanpur

March, 1997

Synopsis

In the theory of elliptic boundary value problems, the phenomenon called "fundamental solution" plays an important role. Fundamental solution and Green's function are very closely related and, infact, they are almost one and the same. The difference between the two is that Green's function is associated with a boundary value problem (i.e. with the differential operator and the boundary conditions), whereas fundamental solution is associated with only the differential operator. Once the Green's function is constructed, the solution is readily represented in terms of the integral of the Green's function and other datas of the problem.

We know that the method of boundary integral equations is the starting point of the today's boundary element method. Here, too, once the fundamental solution is introduced in place of the weighting function, we get a readymade representation of solution in terms of the boundary integrals. To that extent, the fundamental solution is very vital in the boundary element method.

Our work in this thesis is mainly to provide a simple and effective numerical method for singularly perturbed elliptic boundary value problems using the boundary element method. This involves the construction of fundamental solution in a very special way so as the boundary element method suits to singular perturbation problems.

In Chapter two, we start with Helmholtz equation involving a large parameter λ. The fundamental solution of this Helmholtz operator is represented in terms of Hankel function. Since in the boundary element formulation, this large parameter appears

only in the fundamental solution, we go for an asymptotic expansion of the Hankel function, where we have taken only the leading term of the expansion. Using this approximate fundamental solution in the boundary element method we demonstrate that the singularly perturbed Helmholtz equation can be efficiently solved. Further, we have obtained the bounds for the error due the numerical integrations involved in the method and show that this bound tends to zero as $\lambda \to \infty$.

Chapter three, extends this idea to a rather more general linear singularly perturbed boundary value problems of elliptic type in two dimensions. This chapter is diveded mainly into two parts, where we discuss two different linear operators and show that, it is equivalent to considering the general linear problems of singular perturbation type. Here again we construct the fundamental solution for these operators. In this process, we make use of the already availabe fundamental solution of the Helmholtz operator. That is, this fundamental solution is extended in a classical procedure, which involves solving the corresponding Fredholm integral equation. This is done by successive approximation technique, and also shown that the resultant series is convergent for the large parameter λ. An approximation procedure using the dual reciprocity method is presented to construct and evaluate the fundamental solution in an algorithmic way. Finally, we have performed some numerical experiments to show the efficiency of the method.

In chapter four, we propose an iterative technique via linearization for non-linear singularly perturbed elliptic boundary value problems, wherein the linearized problems are solved by the approach discussed in the earlier chapters. The convergence of the iterative technique is proved for very large λ, and the computational experiments exhibits the power and simplicity of the method even for non-linear problems.

Chapter five discusses an application of the method to a physical model in the Semiconductor Physics and which happens to fall under the category of the singular perturbation problems. That is, by proper scaling this non-linear coupled system is

viewed as a singularly perturbed one. Again, via linearization, we decouple the system and solve it iteratively. The linearised problems are solved by boundary element method which uses the fundamental solution constructed in the earlier chapters. Computational results are comparable with already existing ones.

Acknowledgment

I am greatly indebted to my thesis supervisor Professor M. K. Kadalbajoo who introduced me to the field of singular perturbations. His valuable suggestions and critical comments were of great help throughout my thesis work. I thank him for his care and practical advice in several aspects of my life.

I wish to thank all the members of faculty of the department of Mathematics, IIT Kanpur, who have taught me, and encouraged me during my years of stay at IIT K. Particularly, I wish to record my gratitude to Prof. P. C. Das, Prof. O. P. Kapoor, Prof. Manjul Gupta, Prof. Raghavendra, Prof. Shunmughraj, and Prof. Rathish Kumar. My special thanks are due to Prof. K. Sri Ram of Dept. of Nuclear Engg. IIT Kanpur for his words of motivation, and encouragement from time to time.

There are many people who helped me at one time or the other all of whom I cannot list out. I am very privileged to have friends like Drs.Ilango, Ponds, Sampath, Govind, KeenaMoona, SRP, RPPandian, Kasi, Immie, Sokku, Tamil, Punns, Elango, Durai and Messrs. Jeebu, Srini, Anbu, Sivaguru, Pereeps, Rajesh, Kuppi, Ravi, Palani, Venki, Venkatesh(NSI), Murali, Vivek, Sairam, Sivakumar, Ravikumar, Jeyaraj, Kumaresh, Swami, Suresh, Veera, Muthu, Ananth, Subash, Saravanan, Senthan, Johnson, Srini(Hall-V), Sivakumar, Balaji, Kalyan, Ananth, Navanee, Uday, Kali, Saravanan and Sriram. Without the help of these people at my difficult times, my life here would have been miserable.

It gives me a great pleasure to remember my association with my colleagues Dr.CSRao

and Messrs. Sontakke. Dipak, Joydeep, Appaji, Misra, Murthy, SBRao and Magesh.
I thank all the staff of the Dept. of Mathematics, for their cooperation and for providing a good atmosphere.
I must acknowledge the love and care showered by Mrs.Elangovan, Mrs.Punniyamurthy, Mrs.Rajesh, Mrs.Venki and Mrs.Anbu. I thank them all for providing me the much needed family atmosphere which I would have missed otherwise.
The blessings of my parents and the good wishes of my uncles, sisters and other family members have been an invaluable source of inspiration for me.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 General Introduction of the Problem

In this dissertation, we propose some numerical techniques for solving singularly perturbed boundary value problems of elliptic type. We start with linear problems and the extend our work to non-linear ones. The second order elliptic linear singular perturbation problems can in general be described as follows.

$$
\begin{aligned}
Lu \equiv L_2 u + \lambda^2 L_1 u &= \lambda^2 f(x,y) \quad \text{in} \quad \Omega; \quad 1 \ll \lambda < \infty && (1.1)\\
u &= \bar{u} \quad \text{on} \quad \Gamma_1 && (1.2)\\
q\left(= \frac{\partial u_m}{\partial n}\right) &= \bar{q} \quad \text{on} \quad \Gamma_2 && (1.3)
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \qquad L_2 &\equiv \Delta + a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} + c_1\\
L_1 &\equiv a_2 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + c_2\\
\Gamma_1 + \Gamma_2 &= \Gamma; \quad \text{the boundary of} \quad \Omega
\end{aligned}
$$

Ω is a bounded domain in R^2.

It is assumed that the given data $f, a_1, b_1, c_1, a_2, b_2$ and c_2 are sufficiently smooth functions in Ω

The condition for the uniqueness of the solution of the above boundary value problem $c_1 + \lambda c_2 \leq 0$ is assumed to be satisfied in Ω

Wasow[65] writes that

> *whenever in a differential equation the coefficients of the terms of the highest order are small by comparison with the coefficients of the other terms, the solution of the problem can be expected to show irregularities owing to the non-uniform dependence on the coefficients of the differential equation*

Let us indicate, in detail, why this irregularities happen. By setting $\varepsilon = 1/\lambda$, the boundary value problem (1.1)-(1.3) can be written as

$$Lu \equiv \varepsilon^2 L_2 u + L_1 u \quad = \quad f(x,y) \quad \text{in} \quad \Omega; \quad 0 < \varepsilon \ll 1 \tag{1.4}$$

$$u \quad = \quad \bar{u} \quad \text{on} \quad \Gamma_1 \tag{1.5}$$

$$q\left(= \frac{\partial u_m}{\partial n}\right) \quad = \quad \bar{q} \quad \text{on} \quad \Gamma_2 \tag{1.6}$$

In order to solve this boundary value problem, one will be tempted to ignore the term "$\varepsilon^2 L_2 u$", as ε is very small. Then it will reduce (1.4) to

$$L_1 u = f(x,y) \quad \text{in} \quad \Omega \tag{1.7}$$

This equation will be called the 'reduced equation'. Since it is a first order equation, the solution in general will not satisfy the boundary conditions (1.5)-(1.6) on whole Γ. That is, there will be a portion Γ^* of Γ, where the solution of (1.7) will not satisfy the boundary condition, which means that in the neighborhood of Γ^* in Ω, the solution of (1.7) is not a good approximation. This forces us to think of some alternative way

to take some corrective measures in the neighborhood of Γ^*. This involves the construction of the so-called 'boundary layer terms', which are asymptotically equivalent to zero everywhere in Ω except for a small neighborhood of Γ^*. Naturally, we call this neighborhood of Γ^*, the 'boundary layer' region. This construction of asymptotic approximation of the 'boundary layer terms' of the solution is usually the most difficult and most interesting part of the the analysis of singular perturbation problems.

Singular Perturbation Probation problem can still be explained in a simple way. Consider a boundary value problem P_ε depending on a small positive parameter ε. Under some conditions, one can construct the solution u_ε of P_ε by the method of perturbation - i.e. as a power series in ε with the first term being the solution of the reduced equation (1.7). For small values of ε the series cannot often be presumed to converge uniformly in the entire domain Ω. When such an expansion converges as $\varepsilon \to 0$, then the problem P_ε is called *regular perturbation problem.* On the other hand, when u_ε does not have a uniform limit in Ω as $\varepsilon \to 0$, then this straightforward perturbation method fails, and as a consequence one may miscalculate or even lose essential results. In this case, we call P_ε a *singular perturbation problem.* Generally, when the small parameter is multiplied with the highest order terms of the differential equation, as in (1.4), then it happens to be singular perturbation problem. One can define a singular perturbation problem as a one in which no single asymptotic expansion is uniformly valid throughout the the domain Ω, as $\varepsilon \to 0$.

1.2 Asymptotic Analysis of Singular Perturbation problems

Singular Perturbation is now a maturing mathematical subject with a fairly long history and strong promise for continued important applications throughout Science and Engineering. Though the basic intuitive ideas involving local patching of solutions can

be found in early work by Laplace, Kirchoff and others, Prandtl's[55] paper at the 1904 Leipzig mathematical Congress began the study of the fluid dynamical boundary layers by analyzing viscous incompressible flow past an object as the Reynolds number, Re, becomes infinite.

The Navier-Stokes equations, in relation to the Reynolds number, have become one of the most striking examples of singular perturbations, leading to the idea of boundary layers, due to Prandtl. Right afterwards, many fluid dynamicists have got interested and started working on it. However, Friedrichs and Wasow seem to have been the first mathematicians to initiate the mathematical investigation of asymptotic solution of singularly perturbed boundary value problems.Their work was motivated by an analysis of the edge effect for buckled plates. They are the first one to use the term "Singular Perturbation" in the title of[24].

Soon afterwards, Levinson[45] began the study of wide spectrum of important topics in singular perturbations and made intuitive contribution. Mainly, his work on asymptotic properties was based on a maximum principal that can be derived for the differential equation (1.1). Later, Visik and Lysternik[64] simplified Levinson's work, based on norm estimates.

Beginning around 1950, fluid dynamist solved many interesting physical problems, such as the Linoleum-rolling problem (Carrier[10]) and low-Reynolds-number flow past bodies. At Caltech's Guggenheim Aeronautical Laboratory, Lagerstrom, Cole, Latta, Van Dyke[62], Kaplaun and others became equally involved in asymptotic expansion procedures for more general singular perturbation problems,

An oversimplified matching procedure was presented in the book of Van Dyke[62]. The straightforward recipe he provided made it easy for tremendous variety of scientists to learn the rudiments of matching and apply it to important problems in their own disciplines. The basic idea, much as in Friedrichs' earlier work, involved an asymptotic

match of the inner and outer expansions at the edge of the boundary-layer, where they should both be appropriate.

Cole[11] stressed limit process expansions and two timing in a context for broader than fluid mechanics. Indeed, the results obtained through matching generally coincided with those known through the interactive folklore of the various fields. Eckhaus and DeJager[20] have presented a more complete theory of singular perturbation problems for second order linear differential equations of elliptic type.

By 1970, courses in perturbation methods became common in Science, Engineering and Applied Mathematics departments, and inevitably a string of textbooks and high-level monographs began to appear. They include Nayfey[51], Van Dyke[62], Bellman[4], Eckhaus[18, 19], Eckhaus & DeJager[20], O'Malley[52], Hemker[31], Kevorkian & Cole[41], Kaplaun[39], Meyer & Parter[46], Erdélyi[22], Ames[1], Hughes[21]

1.3 Numerical Analysis of Singular Perturbation Problems

The advent of sophisticated computing machines has opened up new areas in almost every field and singular perturbation problem is no exception.

When an asymptotic analysis is valid and a few terms in a asymptotic expansion describe the solution sufficiently accurately, one usually can rely on standard techniques to obtain the solutions. However, when an asymptotic analysis is difficult to handle or performs badly - and this can happen very often considering the complexities occur in the practical problems - one then asks for a numerical method to solve the singular perturbation problem. In a sense, numerical methods are intended for broad classes of problem and are intended to minimize demands upon the problem solver; where as, the asymptotic methods treat comparatively restricted class of problems and require the problem solver to have some understanding of the behavior of the solutions expected.

With the availability of computers, the interest in the area of numerical analysis of singular perturbation problems is increasing among the applied mathematicians. Numerical treatment of singular perturbation problem has always been far from trivial, because of the boundary-layer behavior of the solutions, i.e. if the solution of a perturbation problem exhibits rapid variations within some small region whose extent tends to zero as the small parameter $\varepsilon \to 0$, then the solution is said to posses boundary layer behavior.

It is indeed a big challenge to provide a numerical method to a singular perturbation problem and the numerical analysts have accepted it. This is very evident from the fact that there already exists a vast literature in the field of numerical analysis of singular perturbation problems (for instance, see the survey article by Kadalbajoo & Reddy[38]). However, all these people have mostly confined themselves to only ordinary differential equations and not many research work is done on numerical analysis of singularly perturbed partial differential equations. Of course, we do find research articles here and there in this direction but, overall, we feel that it has received very little attention.

In the conference on Numerical Analysis of singular perturbation problems held at the university of Nijmegen, The Netherlands, there are about six papers that talk about partial differential equations. The first one is Brandt[7]. The approach he has proposed is to first discretize the given singularly perturbed boundary value problem using the finite-element or finite-difference schemes and then use the multi grid adaptive technique. This process automatically takes care of boundary layers. It is shown that the global error decreases exponentially as a function of overall computational work, in a uniform rate independent of the magnitude (ε) of the singular perturbation terms.

Griffiths[29] considers

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - k\frac{\partial u}{\partial x}; \quad (x,t) \in (0,1) \times (0,1) \quad \text{and} \quad k/\varepsilon \gg 1 \tag{1.8}$$

Using the semi-discrete generalized Galerkin method the problem is converted into a system of ordinary differential equations. Analysis of this system has been carried out from the point of view of phase errors, decay rates and the presence of otherwise unphysical oscillations in the solutions.

Veldhuizen[63] first develops a method to solve a first order ordinary differential equation using the finite element space consist of piecewise quadratics and then extend it to a particular type of singularly perturbed second order elliptic boundary value problem.

Axelsson and Gustaffson[2] have presented an interactive method that is essentially based on approximate factorization of the matrix in combination with a minimal residual conjugate procedure to solve the resultant linear systems that is obtained from the discretization of a singularly perturbed second order elliptic boundary value problems.

Baranger[3] has come with a very interesting method that numerically calculates the thickness of the boundary layer. His work was motivated from the result that the solution of the 2-D boundary value problem

$$\begin{aligned} -\varepsilon \Delta u_\varepsilon + u_\varepsilon &= u \quad \text{in} \quad \Omega \\ u_\varepsilon|_\Gamma &= 0 \end{aligned}$$

has an asymptotic expansion of the form

$$u_\varepsilon = u + \exp\left(\frac{-\mu\ d(x,\Gamma)}{\sqrt{\varepsilon}}\right) + O(\varepsilon) \tag{1.9}$$

where Γ is taken to be a unit square and $d(x,\Gamma)$ is the distance from $x \in \Omega$ to Γ. This leads to

$$d(x.\Gamma) = \frac{\sqrt{\varepsilon}}{\mu} log(c\varepsilon)^{-1} = O(\varepsilon^{1/2} log(\varepsilon)^{-1}) \tag{1.10}$$

But since it imposes strong assumptions on Γ (going from C^1 to C^6!) they propose a method to define the thickness of boundary layer based on L^2 estimates outside the

boundary layer which is valid for a Lipschitzian boundary for second order operators with L^∞ coefficients.

Inayat Noor and Aslam Noor[34] prove that by an iterative scheme obtained as a projection of a real Hilbert space into a convex subset the solution of variational inequalities can be found. The asymptotic solution obtained by this iterative method does converge strongly to the exact solution. And it is shown that this method can be extended to a class of singularly perturbed boundary value problems by satisfying some extra conditions.

Jordan[37] in one chapter of his Ph.D. thesis, propose a numerical method for an elliptic boundary value problem of the form

$$\begin{aligned} \left[-\varepsilon\Delta + \frac{\partial}{\partial y} + c(x,y)\right] u &= -f \quad \text{in} \quad \Omega; \\ u &= g(x,y) \end{aligned}$$

where $\Omega = \{(x,y) \in \mathrm{R}^2 : \quad |x-y| < 1\}$.

The method is based on the asymptotic behavior of the solution; i.e., the solution admits an asymptotic expansion of the form

$$u_\varepsilon = U + V + Z$$

where U is the solution of the reduced problem. V is the correction to U in order to meet the remaining conditions and Z is the remainder term that satisfies the estimate $||z|| = o(\varepsilon)$ uniformly in Ω.

Schatz and Wahlbin[59] investigate the behavior of the Galerkin finite element method without any modifications, on a singularly perturbed second order elliptic boundary value problem. Some non-linear elliptic problem also has been discussed and finally some numerical experiments have been carried out.

Ramanujan and Sunanda Kumari[56] propose the method of lines(MOL) for a non-linear elliptic boundary value problem. First, it is shown that the ordinary method of lines doesn't give satisfactory results for singularly perturbed non-linear boundary value problem. Then the modified method is proposed and prove its convergence.

Goldstein has published a string of papers[28, 25–27] on Singularly perturbed (initial and) boundary value problems. His work essentially presents a estimation of the condition number $k({L_h^1}^{-1}L_h^2)$ in terms of the large parameter K and K_1 where L_h^1 and L_h^2 are finite element discretization of two singularly perturbed elliptic operators or order one and two respectively and h is the discretization parameter. Shown how to construct the preconditioning operators such that the condition number of the systems is bounded as the number of unknowns increases and, at worst, grows slowly with K and K_1 increase. Numerical experiments have been performed on elliptic and parabolic convection-diffusion models and two dimensional Helmholtz problems.

Boglaev[6] has presented parallel iterative algorithms via domain decomposition for singularly perturbed nonlinear problems. The domain decomposition is done according to the singular perturbation character of the problems.

1.4 On Boundary Element Method

Integral equation techniques in boundary value problems have been around for a very long time. As far back as 1903, Fredholm used discretized integral equations in potential problems which formed the basis for the "indirect" boundary element approach. This approach is referred to as indirect because it uses fictitious density functions or sources that have no physical meaning but can be used to calculate physical quantities such as displacements and stress. An integral equation relating boundary values of displacements and tractions had been established by Somigliana in 1886. Somiglgna's identity forms the backbone of the "direct" boundary element formulation.

Several books and papers on integral equations in potential and elasticity theory have appeared regularly by distinguished mathematicians such as Kellog[40] in 1929, Muskhelishvili[50] in 1953, Mikhlin[47] in 1957 and Kupradze[42] 1965. The integral formulations, however, were solved by analytical procedures that limited their applications to simple problems. Until the sixties, there was no major work that extended the range of problems that can be solved by integral equations nor considered devising numerical algorithms for solving integral equations.

In the early sixties high speed digital computers and numerical techniques started to find their way into engineering applications. In particular, the finite element method attached a great deal of interest and demonstrated how a wide range of complex engineering problems can be solved using the computer with an impressive accuracy.

The main breakthrough in boundary integral solutions came in 1963 when two classic papers were published by Jaswon[35] and Symm[60]. Their approach consisted of discretizing the integral equations of two-dimensional potential problems governed by Laplace equation into straight line elements over which the potential functions are assumed constant over each element. The elements were described in terms of nodal points and the integrations performed using Simpson's rule except for some singular integrals which were integrated analytically. Jaswon and Symm's approach can be classified as "semi-direct" because the functions used to formulate the problem are not fictitious and can be differentiated or integrated to calculate quantities.

Other similar integral equation approaches were adopted by Jaswon and Ponter[36] for torsion problems involving shafts with different regular cross-sections. Hess and Smith[32] for potential flow problems around arbitrary shapes and Harrington *et al*[30] for two-dimensional electrical engineering problems.

The first paper to use the direct approach of using displacements and tractions in an integral equation applicable over the boundary was published by Rizzo in 1967.

Rizzo's work was very original in that it was the first to exploit the strong analogy between potential theory and classical elasticity theory and devise a numerical approach of solving th problem. He used straight line elements to discretize the boundary where the functions are assumed constant over each element. Simpson's rule was used for all but the singular integrals.

The extension of the direct integral equation approach to three dimensional problems by Cruse[13] in 1969 followed a very similar formulation to Rizzo's works except that the surface was discretized into flat triangular elements with the displacements and tractions assumed constant over each element. Cruse's work was continued in two further publications to include practical three-dimensional applications and comparison with finite element method (Cruse[15] in 1973) and using more sophisticated elements by allowing the variables to change linearly over each element (Cruse[16] in 1974).

During that early period of development (from 1967 to 1972), the integral equation formulations were extended to include non-homogeneous problems containing inclusions (Rizzo and Shippy[58] in 1968), anisotrophic materials (Cruse and Van Buren[17] in 1971 and Cruse[14] in 1972). These early publications were crucial because they provided the firm foundation for further boundary element developments and demonstrated that the boundary element approach was a very powerful and accurate numerical technique that should be taken seriously.

Many of the algorithms and numerical modelling methods used in finite element formulations also found their way into boundary element formulations, such as the concept of higher-order elements, by using quadratic shape functions. These elements were first used by Lachat[43] in 1975 and improved further by Lachat and Watson[44] in 1976. Other authors, such as Tan and Fenner[61], used isoparametric quadratic elements, where both geometry and variables are allowed to change quadratically over each element, and demonstrated the high resolution of stress obtained in three-dimensional

problems.

Since the early seventies the boundary integral equation approach has continued to develop at a fast pace and has been extended to include a very wide range of continuum mechanics including advanced non-linear applications. Over the years, the integral formulations have been referred to as either the boundary integral equation method or the boundary element method.

Comparison of the features of the boundary element method with finite element method can be summarized as follows.

- Less date preparation time. This is a direct result of the reduction of dimensionality by one. Thus the analyst's time required for data preparation for a given problem should be greatly reduced. Furthermore, subsequent changes in meshes are made easier. This advantage is particularly important in problems where re-meshing is required.

- Less computer time and storage. For the same level of accuracy, the boundary element method uses a lesser number of nodes and elements. Since the level of approximation in the boundary element solution is confined to the only boundaries, boundary element meshes should not be compared to the finite element meshes with the internal points removed.

- Less unwanted information. In most engineering problems, the 'worst' situations (such as fracture, stress concentration and thermal shock) usually occur on the surface. In many design codes and engineering practices, the analyst is usually only concerned with what happens in the worst situation. Thus modelling an entire three-dimensional complex body with finite element and calculating stress at every nodal point is very inefficient because only a few of these stress values

will be incorporated in the design analysis. Therefore, using boundary elements is a much more efficient use of computing resources. Furthermore, since internal points in boundary element solutions are optional, the user can focus on a particular interior region rather than the whole interior.

More features of the Boundary Element Method can be found in Brebbia[8, 9].

Of course, there appears to be some disadvantages in boundary element method, one of which is that the solution matrix resulting from the boundary element formulations in unsymmetric and fully populated with non-zero coefficients. However, this is not a serious disadvantage because to obtain the same level of accuracy as the finite element solutions, the boundary element method needs only a relatively modest number of nodes and elements.

1.5 Summary of the thesis

Our work in this thesis is mainly to provide a simple and effective numerical method for singularly perturbed elliptic boundary value problems using the boundary element method. This involves the construction of fundamental solution in a very special way so as the boundary element method suits to singular perturbation problems.

In Chapter two, we start with a singularly perturbed Helmholtz equation. That is, the equation involving a large parameter λ. The fundamental solution of this Helmholtz operator is represented in terms of Hankel function. Since λ is very large, we go for an asymptotic expansion of the Hankel function, where we have taken only the principle term in the expansion. Using this approximate fundamental solution in the boundary element method we demonstrate that the approach is very effective for singularly perturbed boundary value problems.

Chapter three, extends this idea to a rather more general linear singularly perturbed boundary value problems of elliptic type in two dimensions. This chapter is divided mainly into two parts, where we discuss two different forms of the equation (1.1). In the construction of the fundamental solution for these linear operators, we make use of the already available fundamental solution of the Helmholtz operator. That is, this fundamental solution is extended in a classical procedure, which involves solving the corresponding Fredholm integral equation. This we do by successive approximation technique, and show that the resultant series is convergent despite the presence of the large parameter λ. An approximate procedure is presented to construct and evaluate the fundamental solution in an algorithmic way and subsequently performed some numerical experiments to show the efficiency of the method.

In chapter four, we propose an iterative technique for non-linear singularly perturbed elliptic boundary value problems, wherein the linearized problems are solved by the approach given in chapter three. The convergence of the iterative technique is proved for very large λs, and the computational experiments exhibits the power and simplicity of the method even for non-linear problems.

Chapter five discusses an application of the method to the problems in the Semiconductor Physics. Here we deal with a singularly perturbed non-linear systems and the iterative technique proposed in chapter four is extended to this systems. Computational results are comparable with already existing ones.

Chapter 2

The Method for Singularly Perturbed Helmholtz equation

2.1 Introduction

To begin with, we take-up the Helmholtz equation involving a large parameter and propose to apply our technique to this boundary value problem. As we said earlier, the role of fundamental solution is very important in the boundary element method. Keeping the large parameter in mind, we take the asymptotic expansion of the fundamental solution of the Helmholtz operator and use it in the boundary element method. Error bounds of the numerical integrations involved in the method have been obtained along with the computational results.

2.2 The Problem

We consider the boundary value problem

$$-\varepsilon^2 \Delta u + u = 0 \quad \text{in} \quad \Omega \qquad 0 < \varepsilon \ll 1 \tag{2.1}$$

$$u = \bar{u} \quad \text{on} \quad \Gamma_1 \tag{2.2}$$

$$q \left(= \frac{\partial u_m}{\partial n} \right) = \bar{q} \quad \text{on} \quad \Gamma_2 \tag{2.3}$$

where the boundary $\Gamma = \Gamma_1 + \Gamma_2$ *and* $\bar{u}, \bar{q}$ are known functions.

Or,equivalently, by setting $\lambda = 1/\varepsilon$, the problem can be restated as

$$\Delta u - \lambda^2 u = 0 \quad \text{in} \quad \Omega \qquad 1 \ll \lambda < \infty \tag{2.4}$$

$$u = \bar{u} \quad \text{on} \quad \Gamma_1 \tag{2.5}$$

$$q\left(= \frac{\partial u_m}{\partial n}\right) = \bar{q} \quad \text{on} \quad \Gamma_2 \tag{2.6}$$

which is the well-known Helmholtz equation.

This is a particular case of the equation (1.1). i.e., we have taken $L_1 = \Delta$ and $L_2 = -I$, where I is the identity operator.

2.3 Fundamental Solution

As we said earlier, the concept of fundamental solutions is closely related with the Green's functions. For a given differential equation

$$Lu = f \tag{2.7}$$

suppose that L is invertible, i.e. there exist an operator L^{-1} such that $L^{-1}L = LL^{-1} = I$, where I is the identity operator. As L is a general differential operator, it is natural to expect L^{-1} to be an integral operator involving a kernel $P(z, \xi)$, where z and ξ are points in Ω. Thus we write

$$u(z) = L^{-1} f(z) = \int -P(z, \xi) f(\xi) d\xi \tag{2.8}$$

Without loss of generality, we have introduced a negative sign in the above integral just for some convenience. Suppose now we operate on both sides of (2.8) with L again.

Then

$$L[u(z)] = f(z) = L\int -P(z,\xi)f(\xi)d\xi \tag{2.9}$$

As L is a differential operator depending on z, we can write L inside the integral.

$$\int LP(z,\xi)f(\xi)d\xi = -f(z) \tag{2.10}$$

which requires that

$$LP(z,\xi) = -\delta(|z-\xi|) \tag{2.11}$$

where δ is the Dirac delta function and $|z-\xi|$ is the distance between z and ξ. And we call this $P(z,\xi)$, the fundamental solution of the operator L. That is, if we replace f by δ in (2.7) then the solution is called the fundamental solution.

In case of Helmholtz operator, the fundamental solution is

$$P(z,\xi) = (i/4)H_0^{(1)}(i\lambda|z-\xi|) \tag{2.12}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero.

2.4 Formulation of the Method

In this section we present the boundary element method in its simplest form, i.e., the (boundary) elements are linear and the shape functions are constant over each element[8, 9].

We integrate equation (2.4) over the domain Ω with a weighting function w and get

$$\begin{aligned}\int_\Omega [\Delta u - \lambda^2 u] w d\Omega &= 0\\ \\ \int_\Omega \Delta u.wd\Omega - \int_\Omega \lambda^2 u.wd\Omega &= 0\end{aligned}$$

By using the Green's second identity

$$\int_\Omega [w.\Delta u - u.\Delta w] d\Omega = \int_\Gamma \left[w.\frac{\partial u_m}{\partial n} - u.\frac{\partial w}{\partial n} \right] d\Gamma$$

We get

$$\int_\Omega \Delta w.u d\Omega + \int_\Gamma w.\frac{\partial u_m}{\partial n} d\Gamma - \int_\Gamma u.\frac{\partial w}{\partial n} d\Gamma - \int_\Omega \lambda^2 u w d\Omega = 0$$

$$\int_\Omega [\Delta w - \lambda^2 w].u d\Omega - \int_\Gamma u.\frac{\partial w}{\partial n} d\Gamma = -\int_\Gamma w.\frac{\partial u_m}{\partial n} d\Gamma \tag{2.13}$$

The weighting function w is chosen as the fundamental solution. i.e. the solution of

$$[\Delta w - \lambda^2 w] = -\delta_z$$

where $\delta_z = \delta(|z - \xi|)$ is the Dirac delta function and z is the fixed point where the solution is required. Note that we treat $\xi = (\xi_1, \xi_2)$ as a variable in Ω. If r is the distance between z and ξ, then we have

$$w(r) \;=\; \frac{i}{4} H_0^{(1)}(i\lambda r)$$

where $i = \sqrt{-1}$. Since λ is very large we use the asymptotic expansion of the Hankel function[49].

$$H_0^{(1)}(i\lambda r) \simeq e^{[i(i\lambda r - \pi/4)]} \left[\frac{2}{\pi i \lambda r} \right]^{(1/2)}$$

where we have taken only the leading term of the expansion. This leads to

$$w(r) = \frac{i}{4} H_0^{(1)}(i\lambda r) \;\simeq\; \frac{i}{4} e^{(-\lambda r - i\pi/4)} \left[\frac{2}{i\pi\lambda r} \right]^{(1/2)}$$

$$= \frac{i}{4} e^{(-i\pi/4)} \frac{\sqrt{2}}{i^{(1/2)} \sqrt{\pi \lambda r}} e^{(-\lambda r)}$$

$$= \frac{e^{(-\lambda r)}}{2.\sqrt{2\pi\lambda r}}$$

and

$$\frac{\partial w}{\partial r} = \frac{-1}{4\sqrt{2\pi\lambda}} \frac{(1+2\lambda r)e^{-\lambda r}}{r^{(3/2)}}$$

Now, equation (2.13) becomes,

$$\begin{aligned} \int_\Omega (-\Delta_z) u d\Omega - \int_\Gamma u \frac{\partial w}{\partial n} d\Gamma &= -\int_\Gamma w \frac{\partial u_m}{\partial n} d\Gamma \\ -u(z) - \int_\Gamma u \frac{\partial w}{\partial n} d\Gamma &= -\int_\Gamma w \frac{\partial u_m}{\partial n} d\Gamma \end{aligned}$$

(By the property of Dirac delta)

$$u(z) + \int_\Gamma u \frac{\partial w}{\partial n} d\Gamma = + \int_\Gamma w \frac{\partial u_m}{\partial n} d\Gamma \tag{2.14}$$

This expression is valid for any arbitrary interior point z of Ω. Since we do not know u on Γ_2 and $(\partial u/\partial n)$ on Γ_1, we seek an expression for u at a boundary point. Let z_i be a boundary point of Γ. Let Γ_ρ be a semi-circle centered at z_i and of radius ρ. Γ' be the remaining part of Γ such that

$$\lim_{\rho \to 0} \Gamma' = \Gamma$$

Considering $\Gamma_\rho + \Gamma'$ as the boundary, the point z_i becomes an interior point. So, we

apply equation (2.14) for this point and take the limit as $\rho \to 0$.

$$\lim_{\rho\to 0}\left[u(z_i) + \int_{\Gamma'+\Gamma_\rho} u\frac{\partial w}{\partial n}d\Gamma\right] = \lim_{\rho\to 0}\left[\int_{\Gamma'+\Gamma_\rho} w\frac{\partial u_m}{\partial n}d\Gamma\right]$$

$$u(z_i) + \lim_{\rho\to 0}\int_{\Gamma'} u\frac{\partial w}{\partial n}d\Gamma + \lim_{\rho\to 0}\int_{\Gamma_\rho} u\frac{\partial w}{\partial n}d\Gamma = \lim_{\rho\to 0}\int_{\Gamma'} w\frac{\partial u_m}{\partial n}d\Gamma + \lim_{\rho\to 0}\int_{\Gamma_\rho} w\frac{\partial u_m}{\partial n}d\Gamma$$

$$u(z_i) + \int_{\Gamma} u\frac{\partial w}{\partial n}d\Gamma + \lim_{\rho\to 0}\int_{\Gamma_\rho} u\frac{\partial w}{\partial n}d\Gamma = \int_{\Gamma} w\frac{\partial u_m}{\partial n}d\Gamma + \lim_{\rho\to 0}\int_{\Gamma_\rho} w\frac{\partial u_m}{\partial n}d\Gamma \quad (2.15)$$

(2.16)

Consider the second integral term on the left hand side of (2.15).

$$\lim_{\rho\to 0}\int_{\Gamma_\rho} u\frac{\partial w}{\partial n}d\Gamma = \lim_{\rho\to 0}\int_{\Gamma_\rho} u\left(\frac{\partial w}{\partial r}\right)_{r=\rho} d\Gamma$$

Applying mean value theorem for integrals,

$$\lim_{\rho\to 0}\int_{\Gamma_\rho} u\left(\frac{\partial w}{\partial r}\right)_{r=\rho} d\Gamma = \lim_{\rho\to 0} u^*\left(\frac{\partial w}{\partial r}\right)_{r=\rho} \pi\rho$$

$$= \lim_{\rho\to 0} u^*\frac{-(1+2\lambda\rho)e^{(-\lambda\rho)}}{4\sqrt{2\pi\lambda}\ \rho^{3/2}} \pi\rho = 0$$

where u^* is the value of u at some point of Γ_ρ. Similarly, we have

$$\lim_{\rho\to 0}\int_{\Gamma_\rho} (w)_{r=\rho}\frac{\partial u}{\partial n}d\Gamma = \lim_{\rho\to 0}\left(\frac{\partial u}{\partial n}\right)^* \frac{e^{(-\lambda\rho)}}{2\sqrt{2\pi\lambda}\ \rho^{1/2}}\pi\rho = 0$$

Hence equation (2.15) becomes,

$$u(z_i) + \int_\Gamma u \frac{\partial w}{\partial n} d\Gamma = \int_\Gamma w \frac{\partial u_m}{\partial n} d\Gamma \tag{2.17}$$

for any boundary point z_i.

Now, naming the elements of the boundary as $\Gamma_1, \Gamma_2, \ldots.\Gamma_n$,we have $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$. Let the value of u on Γ_j is u_j and that of $(\partial u / \partial n)$ on Γ_j is q_j. Let z_i be the mid-point of the i^{th} element Γ_i. Then $u(z_i) = u_i$.

So, (2.16) can be written as

$$u_i + \int_{\bigcup_j \Gamma_j} u \frac{\partial w}{\partial n} d\Gamma = \int_{\bigcup_j \Gamma_j} w \frac{\partial u_m}{\partial n} d\Gamma$$

$$u_i + \sum_{j=1}^{n} \int_{\Gamma_j} u \frac{\partial w}{\partial n} d\Gamma = \sum_{j=1}^{n} \int_{\Gamma_j} w \frac{\partial u_m}{\partial n} d\Gamma$$

$$u_i + \sum_{j=1}^{n} u_j \int_{\Gamma_j} \frac{\partial w}{\partial n} d\Gamma = \sum_{j=1}^{n} q_j \int_{\Gamma_j} w d\Gamma \tag{2.18}$$

Call $G_{ij} = \int_{\Gamma_j} w d\Gamma$, $\hat{H}_{ij} = \int_{\Gamma_j} (\partial w / \partial n) d\Gamma$, and set

$$\begin{aligned} H_{ij} &= \hat{H}_{ij} && \text{for } i \neq j \\ &= 1 + \hat{H}_{ij} && \text{for } i = j \end{aligned}$$

Then (2.17) can be written as

$$\sum_{j=1}^{n} u_j H_{ij} = \sum_{j=1}^{n} q_j G_{ij} \tag{2.19}$$

Here we know some of u_j's and some of q_j's. i.e. we have exactly n known quantities and n unknowns. Writing the expression (2.18) for i=1(1)n, (i.e.) apply equation (2.16) to every mid-point of the elements $\Gamma_1, \Gamma_2, ...\Gamma_n$ we have the equations in the matrix-vector form

$$\mathbf{H}U = \mathbf{G}Q$$

where

$$\begin{aligned} \mathbf{H} &= [H_{ij}]_{n\times n} \qquad\qquad \mathbf{G} = [G_{ij}]_{n\times n} \\ U &= [u_1, u_2, ...u_n]^T \\ Q &= [q_1, q_2, ...q_n]^T \end{aligned}$$

Rearranging this system with known quantities on one side and the unknowns on the other we get the linear system of equations

$$\mathbf{A}x = F$$

We solve this system by Gaussian Elimination method. Now, we know all the u_j's and q_j's (i.e.) we know both u and $(\partial u/\partial n)$ on Γ.

Now, for an interior point z we have equation (2.14)

$$u(z) + \int_\Gamma u\frac{\partial w}{\partial n}d\Gamma = +\int_\Gamma w\frac{\partial u_m}{\partial n}d\Gamma$$

$$u(z) = -\int_{\Gamma} u\frac{\partial w}{\partial n}d\Gamma + \int_{\Gamma} w\frac{\partial u_m}{\partial n}d\Gamma$$

$$u(z) = -\int_{\cup_j \Gamma_j} u\frac{\partial w}{\partial n}d\Gamma + \int_{\cup_j \Gamma_j} w\frac{\partial u_m}{\partial n}d\Gamma$$

$$u(z) = \sum_{j=1}^{n}[q_j G_{ij} - u_j H_{ij}]$$

We can thus find the solution at any interior point of Ω.

2.5 Error Bounds

In equation (2.17) we do numerical integration to compute the quantities $G_{ij}\&H_{ij}$. We try to obtain some bounds for the truncation error as a function of h, the length of the element, and K where we use K-point Gaussian-quadrature formula. First we define the quantities $r_{ij}^{(1)}\&r_{ij}^{(2)}$ as the distances from z_i, the mid-point of Γ_i, which is the point under consideration, to the starting point and the end point of Γ_j (see Fig-2.1).

Take the transformation,

$$r = \frac{1}{2}[(r_{ij}^{(2)} - r_{ij}^{(1)})t + (r_{ij}^{(2)} + r_{ij}^{(1)})] = \frac{1}{2}(\alpha t + \beta)$$

where, $\alpha = r_{ij}^{(2)} - r_{ij}^{(1)}$ and $\beta = r_{ij}^{(2)} + r_{ij}^{(1)}$.

Now, we have,

$$G_{ij} = \int_{\Gamma_j} w(r)dr$$

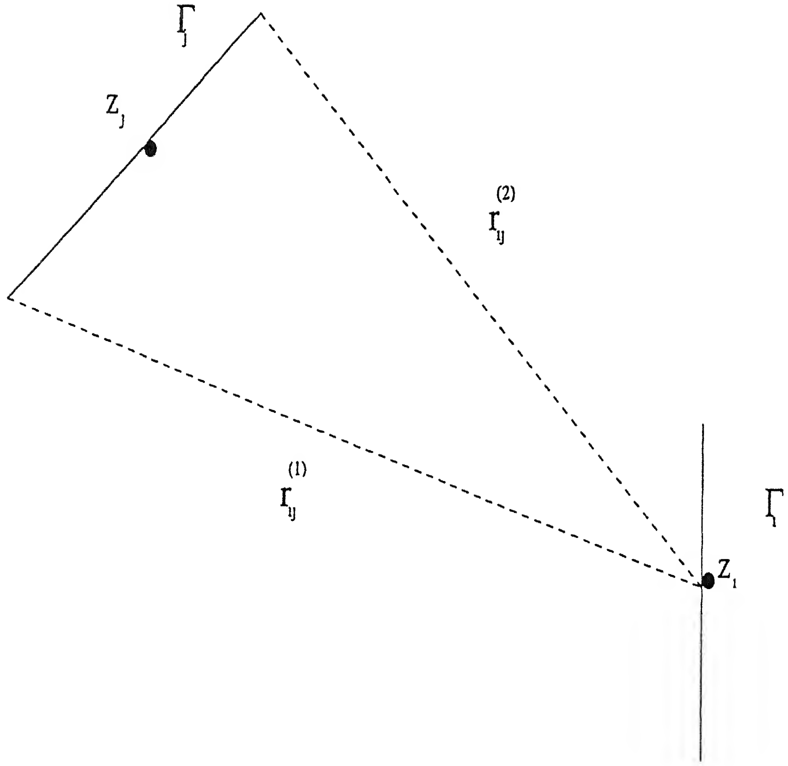


Figure 2.1: $r_{ij}^{(1)}$ and $r_{ij}^{(2)}$ are defined

$$
\begin{aligned}
&= \int_{\Gamma_j} w(r(t))dr(t) \\
&= \int_{-1}^{1} \tilde{w}(t)\ \frac{\alpha}{2} dt \\
&= \frac{\alpha}{2} \sum_{l=1}^{K} w_l\ \tilde{w}(t_l) + E_{ij}
\end{aligned}
$$

where the error term

$$
|E_{ij}| \leq \frac{\alpha}{2^{2K-2}(2K)!} \left(\max_{-1<t<1} \left| \tilde{w}^{(2K)}(t) \right| \right) \tag{2.20}
$$

We have

$$
w(r) = \frac{e^{(-\lambda r)}}{2\sqrt{2\pi\lambda} r^{(1/2)}} \tag{2.21}
$$

$$\tilde{w}(t) \;=\; \frac{\sqrt{2}e^{-\lambda\frac{\alpha t+\beta}{2}}(\alpha t+\beta)^{-1/2}}{2\sqrt{2\pi\lambda}} \tag{2.22}$$

We know that,

$$\tilde{w}^{(2K)}(t) \;=\; \frac{\sqrt{2}}{2\sqrt{2\pi\lambda}}\sum_{l=0}^{2K}\binom{2K}{l}\frac{d^l}{dt^l}\left[e^{\left(-\lambda\frac{(\alpha t+\beta)}{2}\right)}\right]\frac{d^{(2K-l)}}{dt^{2K-l}}\left[(\alpha t+\beta)^{(-1/2)}\right]$$

$$= \frac{\sqrt{2}}{2\sqrt{2\pi\lambda}}\sum_{l=0}^{2K}\binom{2K}{l}\left(\frac{-\lambda}{2}\right)^l\alpha^l e^{-\lambda\frac{(\alpha t+\beta)}{2}}$$
$$\times\left[\alpha^{(2K-l)}\left(\frac{-1}{2}\right)\cdots\left(\frac{-1}{2}-2K+l+1\right)(\alpha t+\beta)^{(-1/2-2K+l)}\right]$$

$$\leq \frac{\alpha^{2K}e^{\left[-\lambda\frac{(\alpha t+\beta)}{2}\right]}}{\sqrt{4\pi\lambda}(\alpha t+\beta)^{(1/2)}}\sum_{l=0}^{2K}\binom{2K}{l}\left(\frac{-\lambda}{2}\right)^l\left(\frac{(1/2)}{(\alpha t+\beta)}\right)^{2K-l}$$

$$= \frac{\alpha^{2K}e^{\left[-\lambda\frac{(\alpha t+\beta)}{2}\right]}}{\sqrt{4\pi\lambda}(\alpha t+\beta)^{(1/2)}}\left(\frac{-\lambda}{2}+\frac{1}{2(\alpha t+\beta)}\right)^{2K}$$

$$= \frac{\alpha^{2K}e^{\left[-\lambda\frac{(\alpha t+\beta)}{2}\right]}\left[1+\lambda(\alpha t+\beta)\right]^{2K}}{\sqrt{4\pi\lambda}(\alpha t+\beta)^{(1/2)+2K}2^{2K}}$$

$$\underset{-1<t<1}{\text{Max}}\left|\tilde{w}^{(2K)}(t)\right| \;\leq\; \left(\frac{|\alpha|^{2K}}{2^{2K}\sqrt{4\pi\lambda}}\right)\frac{\left[\underset{-1<t<1}{\text{Max}}\; e^{-\lambda\left(\frac{\alpha t+\beta}{2}\right)}\right]\left[\underset{-1<t<1}{\text{Max}}\;[1+\lambda(\alpha t+\beta)]^{2K}\right]}{\left[\underset{-1<t<1}{\text{Min}}\;(\alpha t+\beta)^{2K+1/2}\right]}$$

Let $M = \underset{-1<t<1}{\text{Max}}(\alpha t+\beta)$ and $m = \underset{-1<t<1}{\text{Min}}(\alpha t+\beta)$. Since, $\alpha t+\beta = 2r > 0$ always, we have $m > 0$ and $M > 0$. Therefore, we get

$$\underset{-1<t<1}{\text{Max}} \left|\tilde{w}^{(2K)}(t)\right| < \left(\frac{|\alpha|^{2K}}{2^{2K}\sqrt{4\pi\lambda}}\right) \frac{e^{-\lambda m}\,(1+\lambda M^{2K})}{m^{2K+(1/2)}}$$

Suppose h is the (boundary element) discretization parameter, then $\alpha = r_{ij}^{(2)} - r_{ij}^{(1)} < h$, which implies that

$$\underset{-1<t<1}{\text{Max}} \left|\tilde{w}^{(2K)}(t)\right| < \left(\frac{|h|^{2K}}{2^{2K}\sqrt{4\pi\lambda}}\right) \frac{e^{-\lambda m}\,(1+\lambda M^{2K})}{m^{2K+(1/2)}}$$

Substituting this in (2.19), we get

$$|E_{ij}| < \frac{\alpha}{2^{2K-2}(2K)!}\left(\frac{|h|^{2K}}{2^{2K}\sqrt{4\pi\lambda}}\right) \frac{e^{-\lambda m}\,(1+\lambda M^{2K})}{m^{2K+(1/2)}}$$

$$|E_{ij}| < \frac{1}{2^{2K-2}(2K)!}\left(\frac{|h|^{2K+1}}{2^{2K}\sqrt{4\pi\lambda}}\right) \frac{e^{-\lambda m}\,(1+\lambda M^{2K})}{m^{2K+(1/2)}}$$

We note that,

1. $|E_{ij}| \longrightarrow 0$ as $\lambda \to \infty$ for any K and h.

2. $|E_{ij}| \longrightarrow 0$ as $K \to \infty$ for any λ and h.

3. $|E_{ij}| \longrightarrow 0$ as $h \to 0$ for any λ and K.

Following the similar lines, we can show that the truncation error in evaluating H_{ij} tends to zero too.

2.6 Test Example

$$\begin{aligned} -\varepsilon^2\Delta u + u &= 0; \qquad 0 < x, y < 1 \qquad \text{and} \qquad 0 < \varepsilon \ll 1 \\ u &= e^{-x/\varepsilon} + e^{-y/\varepsilon} \text{on the boundary} \end{aligned}$$

We solve the above problem[59] whose exact solution is the boundary condition itself. We have tested the method with 4, 8 and 16 elements of the boundary and to evaluate the matrix elements H_{ij} & G_{ij} we have used 4-point Gaussian quadrature formula. Gaussian Elimination method (with partial pivoting) is used to solve the resulting linear system.

In table-2.1, we have tabulated the sup norm of the error function for various ε's. The much desired aspect here is that the error function decreases with ε. This is because, in using the asymptotic form for the fundamental solution, though we take only the first term of the expansion, the truncation error is of order ε which tends to zero as $\varepsilon \to 0$. Also we note that the solution itself converges to zero as $\varepsilon \to 0$ which coincides with the solution of the reduced problem.

2.7 Conclusion

In this chapter, we have shown that the boundary element method, with the suitable modifications, can be used to solve singularly perturbed boundary value problems where the equation involved, in this case, is Helmholtz equation. We have also obtained the error bounds of the numerical integrations involved in the boundary element method and shown that they tend to zero as the discretization parameter tends to zero for λ large. It is worthy to note that we have used only the simplest boundary element method in the numerical example and can be very well improved by using some higher

Table 2.1: Sup norm of the error function

λ	No. of Boundary Elements			
	4	8	16	32
10^{-1}	$.4632\times10^{-2}$	$.3994\times10^{-2}$	$.1256\times10^{-2}$	$.1113\times10^{-2}$
10^{-2}	$.8241\times10^{-7}$	$.8239\times10^{-7}$	$.8230\times10^{-7}$	$.8226\times10^{-7}$
10^{-3}	$.7411\times10^{-20}$	$.7411\times10^{-20}$	$.7411\times10^{-20}$	$.7411\times10^{-20}$
10^{-4}	$.1931\times10^{-106}$	$.1931\times10^{-106}$	$.1931\times10^{-106}$	$.1931\times10^{-106}$

order elements. The method is very simple and does not need any knowledge about the behavior of the solution.

Chapter 3

Extension of the Method to General Linear Problems

3.1 Introduction

Our aim in this chapter is to present a method for solving a singularly perturbed general linear elliptic problem (1.1) by extending the technique used in the second chapter. First, we shall discuss two different forms of equation (1.1) and then show that the approach can be used to any general linear elliptic problems of the form (1.1). Again, the approach will be to construct the fundamental solution in a suitable way and show the effectiveness of the method for singularly perturbed problems of general linear type. Throughout this chapter, a, b, c and f are assumed to be smooth functions in Ω. We show that the construction of the fundamental solution for the operator L in (1.1), can be done by considering the following problems.

Problem 1

$$L'u \equiv \left[\Delta + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\right]u - \lambda^2 u = f \quad \text{in} \quad \Omega; \qquad 1 \ll \lambda < \infty \tag{3.1}$$

$$u = \bar{u} \quad \text{in} \quad \Gamma_1 \tag{3.2}$$

$$q \left(= \frac{\partial u_m}{\partial n}\right) = \bar{q} \quad \text{in} \quad \Gamma_2 \tag{3.3}$$

We start with the fundamental solution of the Helmholtz operator, $[\Delta - \lambda^2]$, and then extend it to construct the fundamental solution of L'. The process involves solving the corresponding the Fredholm integral equation which is done by successive approximation method. The resultant series is shown to be convergent for large parameter λ.

Problem 2

$$L''u \equiv \Delta u + \lambda^2 \left[a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c \right] u \quad = \quad \lambda^2 f \quad \text{in} \quad \Omega; \qquad 1 \ll \lambda < \infty \qquad (3.4)$$

with boundary conditions (3.2) and (3.3).

Here the construction procedure is same as in the case of problem 1. The only difference is that the kernel involved in the Fredholm integral equation is different. In this case also, we have shown that the resultant series in solving the Fredholm integral equation converges uniformly in Ω for large λ.

Section 3.2 shows that the fundamental solution of L' can be done from $[\Delta - \lambda^2]$ and section 3.3 gives the fundamental solution for L''. Combining these two, we can see that the fundamental solution can be obtained for the operator L from the fundamental solution of L'' by following the lines of section (3.2).

3.2 Construction of the Fundamental Solution for L'

In this section we construct[12, 57] the fundamental solution for the operator

$$L' \equiv \left[\Delta + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c \right] - \lambda^2$$

The fundamental solution of the operator $[\Delta - \lambda^2]$ is

$$P_0(z-\xi,\xi;\lambda) \simeq \frac{1}{2\sqrt{2\pi\lambda}}\left[\frac{\exp(-\lambda|z-\xi|)}{\sqrt{|z-\xi|}}\right] \tag{3.5}$$

From (3.5) we shall extend to the fundamental solution for the operator L'. In order to do this, let us seek a fundamental solution with a singularity at a point $\xi = (\xi_1, \xi_2) \in \Omega$ in the form

$$P(z,\xi,\lambda) = P_0(z-\xi,\xi,\lambda) + \int_\Omega P_0(z-\eta,\eta,\lambda)h(\eta,\xi,\lambda)d_\eta\Omega \tag{3.6}$$

where $h(\eta,\xi,\lambda)$ is determined such that when $z \neq \xi$, $P(z,\xi,\lambda)$ satisfies $L'P = 0$. Then from (3.6),

$$L'P_0(z-\xi,\xi,\lambda) + L'\int_\Omega P_0(z-\eta,\eta,\lambda)h(\eta,\xi,\lambda)d_\eta\Omega = 0 \tag{3.7}$$

Since the operator L' is with respect to $z = (x,y)$ and the above integral is w.r.to η, (3.7) can be written as

$$L'P_0(z-\xi,\xi,\lambda) + \int_\Omega L'P_0(z-\eta,\eta,\lambda)h(\eta,\xi,\lambda)d_\eta\Omega = 0 \tag{3.8}$$

Now consider
$$\begin{aligned} L'P_0 &= [\Delta - \lambda^2]P_0 + \left[a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\right]P_0 \\ &= -\delta_z(\xi) + k(z,\xi,\lambda) \end{aligned}$$

where $\delta_z(\xi)$ is the Dirac delta function and

$$k(z,\xi,\lambda) = \left[a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\right]P_0(z-\xi,\xi,\lambda) \quad . \tag{3.9}$$

Now, (3.8) becomes.

$$-\delta_z(\xi) + k(z,\xi,\lambda) + \int_\Omega [-\delta_z(\xi) + k(z,\xi,\lambda)]h(\eta,\xi,\lambda)d_\eta\Omega = 0$$
$$-\delta_z(\xi) + k(z,\xi,\lambda) + \int_\Omega -\delta_z(\xi)h(\eta,\xi,\lambda)d_\eta\Omega + \int_\Omega k(z,\xi,\lambda)]h(\eta,\xi,\lambda)d_\eta\Omega = 0$$
$$-\delta_z(\xi) + k(z,\xi,\lambda) - h(z,\xi,\lambda) + \int_\Omega k(z,\xi,\lambda)]h(\eta,\xi,\lambda)d_\eta\Omega = 0$$

Since $\delta_z(\xi) = 0$ for $z \neq \xi$, we have

$$h(z,\xi,\lambda) = k(z,\xi,\lambda) + \int_\Omega k(z,\eta,\lambda)h(\eta,\xi,\lambda)d_\eta\Omega \tag{3.10}$$

The solution of the integral equation (3.10) can be constructed by the successive approximations. Thus for $h(z,\xi,\lambda)$ we have the series

$$h(z,\xi,\lambda) = k(z,\xi,\lambda) + \sum_{\nu=2}^{\infty} k_\nu(z,\xi,\lambda) \tag{3.11}$$

where $k_\nu(z,\xi,\lambda)$ is the νth iteration of the kernel $k(z,\xi,\lambda)$. We show that the above series is convergent w.r.to ξ in the region Ω for sufficiently large λ. From equation(3.9),

$$k(z,\xi,\lambda) = \left[a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\right] \frac{1}{2\sqrt{2\pi\lambda}} \left(\frac{\exp(-\lambda r)}{\sqrt{r}}\right)$$

where $r = |z-\xi| = \sqrt{(x-\xi_1)^2 + (y-\xi_2)^2}$. Then, we have

$$\begin{aligned} k(z,\xi,\lambda) &= \frac{1}{2\sqrt{2\pi\lambda}} \left(\frac{\exp(-\lambda r)}{r^{3/2}}\right) \left[cr - \left(\lambda + \frac{1}{2r}\right)(x-\xi_1)a - \left(\lambda + \frac{1}{2r}\right)(y-\xi_2)b\right] \\ &= \frac{1}{2\sqrt{2\pi\lambda}} \left(\frac{\exp(-\lambda r)}{r^{3/2}}\right) \\ &\qquad \left[cr + \frac{1}{2r}[(x-\xi_1)a + (y-\xi_2)b] - \lambda[(x-\xi_1)a + (y-\xi_2)b]\right] \end{aligned}$$

$$= \frac{1}{2\sqrt{2\pi\lambda}}\left(\frac{\exp(-\lambda r)}{r^2}\right)$$
$$\left[cr^{3/2} + \frac{1}{2\sqrt{r}}[(x-\xi_1)a + (y-\xi_2)b] - \lambda\sqrt{r}[(x-\xi_1)a + (y-\xi_2)b]\right]$$

Using the continuity properties of a, b and c, we can find constants C_1' and C_2' such that

$$\left|cr^{3/2} + \left(\frac{1}{2\sqrt{r}}\right)[(x-\xi_1)a + (y-\xi_2)b]\right| < C_1'$$
$$\left|\sqrt{r}[(x-\xi_1)a + (y-\xi_2)b]\right| < C_2'$$

From the above inequalities, it follows that

$$|k(z,\xi,\lambda)| \leq \frac{1}{\sqrt{\lambda}}\frac{\exp(-\lambda r)}{r^2}(C_1' + \lambda C_2')$$

$$= \frac{1}{\sqrt{\lambda}}\frac{\exp(-\frac{\lambda}{2}r)}{r^2}\left[C_1'\exp\left(-\frac{\lambda}{2}r\right) + \lambda C_2'\exp\left(-\frac{\lambda}{2}r\right)\right]$$

Fix λ_0 sufficiently large. Then for all $\lambda \geq \lambda_0$, we obtain

$$|k(z,\xi,\lambda)| \leq C'\frac{\exp\left(-\frac{\lambda}{2}r\right)}{r^2}$$

or,

$$|k(z,\xi,\lambda)| \leq C'\frac{\exp\left(-\frac{\lambda}{2}|z-\xi|\right)}{|z-\xi|^2} \tag{3.12}$$

where C' is a constant independent of λ. (It can be noticed that for any positive r, $[\lambda\exp(-\frac{\lambda}{2}r)] \to 0$ as $\lambda \to \infty$.)

Now to obtain a bound for the 2nd iteration of the kernel $k(z,\xi,\lambda)$,

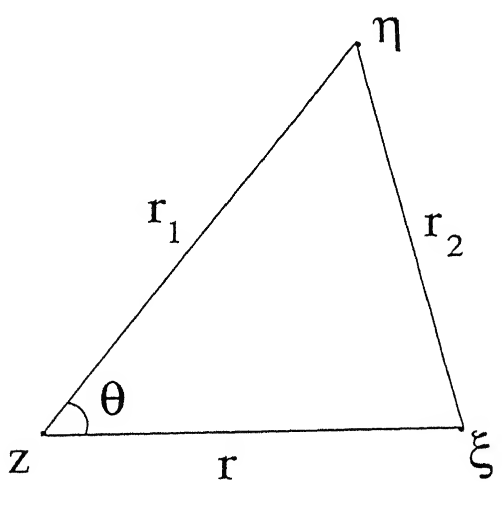
$$|k_2(z,\xi,\lambda)| = \left|\int_\Omega k(z,\eta,\lambda)k(\eta,\xi,\lambda)d_\eta\Omega\right|$$

$$< \int_\Omega C'^2 \frac{\exp\left(\frac{-\lambda}{2}(|z-\eta|+|\eta-\xi|)\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega \qquad \text{(from (3.12))}$$

$$< C'^2 \exp\left(\frac{-\lambda}{4}(|z-\xi|)\right) \int_\Omega \frac{\exp\left(\frac{-\lambda}{4}(|z-\eta|)\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega$$

$$< C'^2 \left[\exp\left(\frac{-\lambda}{4}|z-\xi|\right)\right] \int_\Omega \frac{\exp\left(\frac{-\lambda}{4}|z-\eta|\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega \tag{3.13}$$

To find a bound for the right hand side integral, we transform to polar co-ordinates, with origin at z and take one axis along the vector $z-\xi$. And let $|z-\xi| = r$, $|z-\eta| = r_1$ and $|\eta-\xi| = r_2$.



Then, it follows that

$$\int_\Omega \frac{\exp\left(\frac{-\lambda}{4}|z-\eta|\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega < \int_0^\infty dr_1 \int_0^{2\pi} \frac{\exp(-\lambda r_1)}{r_1^2 r_2^2} r_1^2 \sin\theta \, d\theta$$

$$= \int_0^\infty dr_1 \int_0^{2\pi} \frac{\exp(-\lambda r_1)\sin\theta}{r_1^2 + r^2 - 2rr_1\cos\theta}\, d\theta$$

$$\leq \frac{1}{r}\int_0^\infty \exp(-\lambda rt) \int_0^{2\pi} \left|\frac{\sin\theta}{1+t^2-2t\cos\theta}\right| \,d\theta\, dt$$

$$= \frac{2}{r}\int_0^\infty \exp(-\lambda rt)\frac{1}{2t}\ln\frac{1+t^2+2t}{1+t^2-2t}\, dt$$

$$= \frac{1}{r}\int_0^\infty \exp(-\lambda rt)\frac{1}{t}\ln\frac{1+t}{1-t}\, dt$$

$$< \frac{1}{\lambda r^2}$$

or;

$$\int_\Omega \frac{\exp\left(\frac{-\lambda}{4}|z-\eta|\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega < \frac{1}{\lambda|z-\xi|^2} \tag{3.14}$$

Using (3.14) in (3.13) we get

$$|k_2(z,\xi,\lambda)| < \left(\frac{C'^2}{\lambda}\right)\frac{\exp\left(\frac{-\lambda}{4}|z-\xi|\right)}{|z-\xi|^2} \tag{3.15}$$

Now, we show by induction that for $n \geq 2$,

$$|k_n(z,\xi,\lambda)| < \left(\frac{C'}{\lambda}\right)^{n-1}\frac{C'}{|z-\xi|^2}\exp\left(\frac{-\lambda}{4}|z-\xi|\right) \tag{3.16}$$

Assume that (3.16) is valid for $n = m-1$ for some m, and we show that it is true for $n = m$.

$$
\begin{aligned}
|k_m(z,\xi,\lambda)| &= \left|\int_\Omega k_{m-1}(z,\eta,\lambda)k(\eta,\xi,\lambda)d_\eta\Omega\right| \\
&< \int_\Omega \left(\frac{C'^2}{\lambda}\right)^{m-2} C'^2 \frac{\exp\left(\frac{-\lambda}{4}|z-\eta|\right)\exp\left(\frac{-\lambda}{2}|\eta-\xi|\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega \\
&= \left(\frac{C'^2}{\lambda}\right)^{m-2} C'^2 \int_\Omega \frac{\exp\left(\frac{-\lambda}{4}|z-\eta|\right)\exp\left(\frac{-\lambda}{2}|\eta-\xi|\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega \\
&= \left(\frac{C'^2}{\lambda}\right)^{m-2} C'^2 \int_\Omega \frac{\exp\left(\frac{-\lambda}{4}|z-\eta| + \frac{\lambda}{4}|\eta-\xi|\right)\exp\left(\frac{-\lambda}{4}|\eta-\xi|\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega \\
&\leq \left(\frac{C'^2}{\lambda}\right)^{m-2} C'^2 \exp\left(\frac{-\lambda}{4}|z-\xi|\right) \int_\Omega \frac{\exp\left(\frac{-\lambda}{4}|\eta-\xi|\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega
\end{aligned}
$$

From (3.14), we know that

$$
\int_\Omega \frac{\exp\left(\frac{-\lambda}{4}|\xi-\eta|\right)}{|z-\eta|^2|\eta-\xi|^2} d_\eta\Omega < \frac{1}{\lambda|z-\xi|^2}
$$

This leads to

$$
|k_m(z,\xi,\lambda)| < \left(\frac{C'^2}{\lambda}\right)^{m-1} \frac{C'^2}{|z-\xi|^2} \exp\left(\frac{-\lambda}{4}|z-\xi|\right)
$$

Hence the inequality (3.16). This inequality is valid for all $\lambda \geq \lambda_0$, where λ_0 is sufficiently large. From this the convergence of the (3.11) can be easily obtained.

3.3 Construction of the Fundamental Solution for L''

In this section we construct the fundamental solution for the operator

$$L'' \equiv \left[\Delta + a\lambda^2\frac{\partial u}{\partial x} + b\lambda^2\frac{\partial u}{\partial y} + c\lambda^2\right]$$

$$\text{or,} \qquad L'' \equiv \left[\Delta + a\lambda^2\frac{\partial u}{\partial x} + b\lambda^2\frac{\partial u}{\partial y} + (c+1)\lambda^2\right] - \lambda^2$$

The procedure is same as in the previous section except that the kernel involved in the Fredholm integral equation is different in this case. The fundamental solution of the operator $[\Delta - \lambda^2]$ is given by (3.5).

Following the lines of the previous section, we shall seek the fundamental solution in the form of (3.6). Then, the corresponding Fredholm integral equation is (3.10) but the kernel $k(z, \eta, \lambda)$, in this case, is

$$k(z, \xi, \lambda) = \left[a\lambda^2\frac{\partial u}{\partial x} + b\lambda^2\frac{\partial u}{\partial y} + (c+1)\lambda^2\right] P_0(z - \xi, \xi, \lambda)$$

The solution for $h(z, \xi, \lambda)$ in the integral equation (3.10) is same as (3.11) but the kernel and the corresponding iterated kernels will be given by (3.17).

We show that the series (3.11) corresponding to the kernel (3.15) is convergent w.r.to ξ in the region Ω for sufficiently large λ. From (3.7) and (3.15) we have

$$k(z, \xi, \lambda) = \left[a\lambda^2\frac{\partial u}{\partial x} + b\lambda^2\frac{\partial u}{\partial y} + (c+1)\lambda^2\right] \frac{1}{2\pi\sqrt{\lambda}} \left(\frac{\exp(-\lambda r)}{\sqrt{r}}\right)$$

where $r = |z-\xi| = \sqrt{(x-\xi_1)^2+(y-\xi_2)^2}$. Then, we have

$$\begin{aligned} k(z,\xi,\lambda) &= \frac{1}{2\pi\sqrt{\lambda}}\left(\frac{\exp(-\lambda r)}{r^{3/2}}\right) \\ &\left[(c+1)\lambda^2 r - \left(\lambda+\frac{1}{2r}\right)(x-\xi_1)a\lambda^2 - \left(\lambda+\frac{1}{2r}\right)(y-\xi_2)b\lambda^2\right] \\ &= \frac{1}{2\pi\sqrt{\lambda}}\left(\frac{\exp(-\lambda r)}{r^2}\right) \\ &\left[\left\{(c+1)r^{3/2} - \frac{1}{2\sqrt{r}}[(x-\xi_1)a-(y-\xi_2)b]\right\}\lambda^2 - \{(x-\xi_1)a+(y-\xi_2)b\}\lambda^3\sqrt{r}\right. \end{aligned}$$

Using the continuity properties of a, b and c, we can find constants C_1'' and C_2'' such that

$$\begin{aligned} \left|(c+1)r^{3/2} + \left(\frac{1}{2\sqrt{r}}\right)[(x-\xi_1)a+(y-\xi_2)b]\right| &< C_1'' \\ \left|\sqrt{r}[(x-\xi_1)a+(y-\xi_2)b]\right| &< C_2'' \end{aligned}$$

Substituting these constants in the expression for $k(z,\xi,\lambda)$, we get

$$\begin{aligned} |k(z,\xi,\lambda)| &\leq \frac{1}{\sqrt{\lambda}}\frac{\exp(-\lambda r)}{r^2}(C_1''\lambda^2+\lambda^3 C_2'') \\ &= \frac{1}{\sqrt{\lambda}}\frac{\exp\left(-\frac{\lambda}{2}r\right)}{r^2}\left[C_1''\lambda^2\exp\left(-\frac{\lambda}{2}r\right)+\lambda^3 C_2''\exp\left(-\frac{\lambda}{2}r\right)\right] \end{aligned}$$

As in the previous section, for λ_0 sufficiently large, we get

$$|k(z,\xi,\lambda)| < C''\frac{\exp(-\frac{\lambda}{2}|z-\xi|)}{|z-\xi|^2}.$$

where C'' is a constant independent of λ. Further, in the similar way as in section-3.2, we can show that

$$|k_n(z,\xi,\lambda)| < \left(\frac{C''}{\lambda}\right)^{n-1} \frac{C''}{\lambda|z-\xi|^2} \exp\left(\frac{-\lambda}{4}|z-\xi|\right)$$

From this the convergence of the (3.9) can be easily obtained.

3.4 Approximation of the Fundamental Solution

The fundamental solution is given by the equation (3.6) where $h(\eta,\xi,\lambda)$ is represented by the convergent series (3.10). From (3.16) this series can be written by

$$h(z,\xi,\lambda) = k(z,\xi,\lambda) + \sum_{\nu=2}^{l} k_\nu(z,\xi,\lambda) + O\left(\lambda^{-l}\right) \tag{3.17}$$

for some suitable l. Hence for large λ, this can be written as

$$h(z,\xi,\lambda) \simeq k(z,\xi,\lambda) + \sum_{\nu=2}^{l} k_\nu(z,\xi,\lambda) \tag{3.18}$$

The number of terms in this series could depends on λ. For very large λ, sometimes even $k(z,\xi,\lambda)$ will be a good approximation. To evaluate the right hand side integral in (3.6) we use a slightly modified version of the dual reciprocity method given in Partridge[54] as follows. The approximation for $h(\eta,\xi,\lambda)$ is proposed by

$$h \simeq \sum_{j=1}^{N+L} \phi_j f_j \tag{3.19}$$

where ϕ_j are a set of initially unknown coefficients and the f_j are approximation functions. N is the number of boundary nodes and L is the number of interior nodes in the problem domain. The above approximation is such that h is exact at these $(N+L)$ nodes. The expansion may then be considered as valid over the whole problem domain. Now, we choose a series of functions $\hat{u}_j$ such that

$$[\Delta - \lambda^2]\hat{u}_j = f_j \tag{3.20}$$

Now consider

$$\begin{aligned}
\int_\Omega P_0(z-\eta,\eta,\lambda)h(\eta,\xi,\lambda)d_\eta\Omega &= \int_\Omega P_0(z-\eta,\eta,\lambda)\left[\sum_{j=1}^{N+L}\phi_j f_j\right]d_\eta\Omega \\
&= \int_\Omega P_0\left[\sum_{j=1}^{N+L}\phi_j[\Delta-\lambda^2]\hat{u}_j\right]d_\eta\Omega \\
&= \sum_{j=1}^{N+L}\phi_j\int_\Omega P_0[\Delta-\lambda^2]\hat{u}_j\, d_\eta\Omega \qquad (3.21)
\end{aligned}$$

Since P_0 is the fundamental solution of the operator $[\Delta - \lambda^2]$ we can use the boundary element formulation given in the section 2.4 of chapter 2. From (2.14), (3.21) can be written as

$$\int_\Omega P_0\; h\; d\Omega = \sum_{j=1}^{N+L}\phi_j\left[\hat{u}_{ij} + \sum_{k=1}^{N}\int_{\Gamma_k}(\partial P_0/\partial n)\hat{u}_j d\Gamma - \sum_{k=1}^{N}\int_{\Gamma_k}P_0\;\hat{q}_j d\Gamma\right]$$

Using the notations as given in section 2.4 in chapter 2, we get

$$= \sum_{j=1}^{N+L} \phi_j \left[\hat{u}_{ij} + \sum_{k=1}^{N} H_{ik}\hat{u}_{kj} - \sum_{k=1}^{N} G_{ik}\hat{q}_{kj} \right]$$

$$= \sum_{j=1}^{N+L} \phi_j \left[H\hat{u}_j - G\hat{q}_j \right]$$

This involves discretization of the boundary only. Internal nodes may be defined in the number and at the locations desired by the user; this is generally done at points where it is desirable to know the solutions.

In order to find the vector ϕ, consider (3.19). By taking $(N+L)$ different points, a set of equations like (3.19) is obtained. This may be expressed in matrix-vector form

$$\boldsymbol{h} = \boldsymbol{F}\boldsymbol{\phi} \tag{3.22}$$

where each column of $\boldsymbol{F}$ consists of a vector f_j containing the values of function f_j at $(N+L)$ collocation points. As h is a known function, from (3.22) we have

$$\boldsymbol{\phi} = \boldsymbol{F}^{-1}\boldsymbol{h} \tag{3.23}$$

Thus ϕ will be a known vector.

3.5 Formulation of the method

In this section, we show the formulation of the boundary element method method for the boundary value problem (3.1)-(3.3). The formulation for problem 2 will be similar. We integrate equation (3.1) with a weighting function w,

$$\int_\Omega (Lu)wd\Omega = \int_\Omega fwd\Omega \tag{3.24}$$

$$\int_\Omega (\Delta u)wd\Omega + \int_\Omega a\frac{\partial u}{\partial x}wd\Omega + \int_\Omega b\frac{\partial u}{\partial y}wd\Omega + \int_\Omega cuwd\Omega + \int_\Omega (-\lambda^2)uwd\Omega = \int_\Omega fwd\Omega$$

Using Green's identities,

$$\begin{aligned}\int_\Omega (\Delta w)ud\Omega + \int_\Gamma w\frac{\partial u_m}{\partial n}d\Gamma - \int_\Gamma u\frac{\partial w}{\partial n}d\Gamma - \int_\Omega au\frac{\partial w}{\partial x}d\omega &- \int_\Omega \frac{\partial a}{\partial x}uwd\Omega \\ + \int_\Gamma auwdy - \int_\Omega bu\frac{\partial w}{\partial y}d\Omega - \int_\Omega \frac{\partial b}{\partial y}uwd\Omega &- \int_\Gamma buwdx + \int_\Omega cuwd\Omega \\ &- \int_\Omega \lambda^2 uwd\Omega = \int_\Omega fwd\Omega\end{aligned}$$

$$\begin{aligned}\int_\Omega \left[\Delta w - a\frac{\partial w}{\partial x} - b\frac{\partial w}{\partial y} - \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + c\right) w - \lambda^2 w\right] ud\Omega &+ \int_\Gamma w\frac{\partial u_m}{\partial n}d\Gamma \\ - \int_\Gamma u\frac{\partial w}{\partial n}d\Gamma + \int_\Gamma auwdy - \int_\Gamma buwdx &- \int_\Omega fwd\Omega = 0\end{aligned}$$

$$\int_\Omega (L^* w)ud\Omega - \int_\Gamma \frac{\partial w}{\partial n}ud\Gamma + \int_\Gamma awudy - \int_\Gamma bwudx - \int_\Omega fwd\Omega = -\int_\Gamma w\frac{\partial u_m}{\partial n}d\Gamma \tag{3.25}$$

where L^* is the adjoint operator of L.

$$\text{i.e.}\quad L^* \equiv \left[\Delta - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y} - \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + c\right) - \lambda^2\right]$$

Choosing w as the fundamental solution of the operator L^* with respect to the point 'i', where 'i' is any internal point of Ω where we want to find the solution.

Then (3.25) becomes

$$-u^i - \int_\Gamma \frac{\partial w}{\partial n} u d\Gamma + \int_\Gamma awudy - \int_\Gamma bwudx - \int_\Omega fwd\Omega = -\int_\Gamma w \frac{\partial u_m}{\partial n} d\Gamma \qquad (3.26)$$

The expression (3.26) is valid for any internal point 'i' of Ω and it can be seen that the same expression is valid for a boundary point also.

Now assume that Ω is a polygonal domain and $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$. Further, u and $(\partial u/\partial n)$ are assumed to be constant on each Γ_j. Then for a boundary point 'i', we have

$$-u^i - \int_\Gamma \frac{\partial w}{\partial n} u d\Gamma + \int_\Gamma awudy - \int_\Gamma bwudx - \int_\Omega fwd\Omega = -\int_\Gamma w \frac{\partial u_m}{\partial n} d\Gamma$$

$$-u^i - \int_{\bigcup_j \Gamma_j} \frac{\partial w}{\partial n} u d\Gamma + \int_{\bigcup_j \Gamma_j} awudy - \int_{\bigcup_j \Gamma_j} bwudx - \int_\Omega fwd\Omega = -\int_{\bigcup_j \Gamma_j} w \frac{\partial u_m}{\partial n} d\Gamma$$

$$-u^i - \sum_j \int_{\Gamma_j} \frac{\partial w}{\partial n} u_j d\Gamma + \sum_j \int_{\Gamma_j} awu_j dy - \sum_j \int_{\Gamma_j} bwu_j dx - \int_\Omega fwd\Omega = -\sum_j \int_{\Gamma_j} wq_j d\Gamma$$

$$u^i + \sum_j u_j \left[\int_{\Gamma_j} \frac{\partial w}{\partial n} d\Gamma + \int_{\Gamma_j} awdy - \int_{\Gamma_j} bwdx\right] + \int_\Omega fwd\Omega = \sum_j q_j \left[\int_{\Gamma_j} wd\Gamma\right]$$

$$u^i + \sum_j \hat{H}_{ij} u_j + B_i = \sum_j G_{ij} q_j \tag{3.27}$$

$$\text{where} \quad \hat{H}_{ij} = \left[\int_{\Gamma_j} \frac{\partial w}{\partial n} d\Gamma + \int_{\Gamma_j} awdy - \int_{\Gamma_j} bwdx\right]$$

$$G_{ij} = \int_{\Gamma_j} wd\Gamma \qquad \& \quad B_i = \int_\Omega fwd\Omega.$$

$$\begin{aligned} \text{Set} \quad H_{ij} &= \hat{H}_{ij} \quad \text{for } i \neq j \\ &= 1 + \hat{H}_{ij} \quad \text{for } i = j \end{aligned}$$

Then (3.27) gives

$$\sum_j^n H_{ij} u_j + B_i = \sum_j^n G_{ij} q_j \tag{3.28}$$

We know only some of the u_j's & q_j's. Keeping the known quantities on one side and the unknowns on the other we can write (3.28) in the matrix vector form

$$Ax = g \tag{3.29}$$

Solving this system of linear equations we will know all the u_j's and q_j's. Then again from (3.26) we have

$$u^i = -\sum_j \hat{H}_{ij} u_j - B_i + \sum_j G_{ij} q_j$$

Thus we can find the solution at any internal point of Ω

3.6 Numerical Examples

Example 1

$$\begin{aligned} \Delta u + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - \lambda^2 u &= f\,, \qquad 0 < x, y < 1 \quad \& \quad 1 \ll \lambda < \infty \\ u &= \exp(-\lambda x) + \exp(-\lambda y) \quad \text{on the boundary.} \end{aligned}$$

Here $f = -\lambda[x\exp(-\lambda x) + y\exp(-\lambda y)]$. This problem has the obvious exact solution. To evaluate B_i, we discretize the unit square into 16 square elements and then integrate f against the fundamental solution over each element using Gaussian quadrature formula and then take the sum of all these terms. One can avoid domain integration here by directly using the dual reciprocity method [4]. To evaluate the matrix elements H_{ij} & G_{ij} we have used 4-point Gaussian quadrature formula. Gauss Elimination method (with partial pivoting) is used to solve the resulting linear system.

To find out the solutions at the internal points, we define a grid of points (x_k, y_l) where $x_k = k/K+1; k = 1(1)K$ and $y_l = l/K+1; l = 1(1)K$ and $K > \lambda$ so that we consider the solution at the points of the boundary layer region also. We call the approximate solution at the point (x_k, y_l) as U_{kl} and define $err(k,l) = |u_{kl} - U_{kl}|$. Then the maximum of this error function $\underset{k,l}{\text{Max}}\ [err(k,l)]$ which is tabulated for various λ's and different number of boundary elements in table 3.1. As we use the asymptotic form for the fundamental solution we see that as λ increases, the error decreases. And that we could achieve in just 4 elements! Also we note that the solution itself converges to zero

as $\lambda \to \infty$ which coincides with the solution of the reduced problem.

Table 3.1: Sup norm of the error function in Ω of Problem1.

λ	No. of Boundary Elements			
	4	8	16	32
2	1.952418	1.203472	.791541	.483652
10	1.046732	0.842572	.197314	$.786261\times10^{-1}$
20	0.964321	0.277501	$.960891\times10^{-1}$	$.342605\times10^{-1}$
100	$.181534\times10^{-1}$	$.180921\times10^{-1}$	$.180701\times10^{-1}$	$.179072\times10^{-1}$
200	$.921938\times10^{-3}$	$.907138\times10^{-3}$	$.894832\times10^{-3}$	$.855028\times10^{-3}$
1000	$.628398\times10^{-20}$	$.593045\times10^{-20}$	$.563486\times10^{-20}$	$.552394\times10^{-20}$
2000	$.732947\times10^{-39}$	$.709230\times10^{-39}$	$.690274\times10^{-39}$	$.690186\times10^{-39}$

Example 2

$$\begin{aligned}
\Delta u + \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right] - \lambda^2 u &= f\,, \quad 0 < x, y < \pi \quad \& \quad 1 \ll \lambda < \infty, \\
\frac{\partial}{\partial x} u(0,y) = -M\cos(y/2) & \qquad \frac{\partial}{\partial x} u(\pi, y) = 0, \\
\frac{\partial}{\partial y} u(x,0) = -\lambda, & \qquad \text{and} \qquad u(x,\pi) = 0.
\end{aligned}$$

Here f is given by

$$f \;=\; \lambda^2 \exp(-\lambda\pi) - [M\exp(-Mx)\cos(y/2) + (1/2)\exp(-Mx)\sin(y/2) - \lambda\exp(-\lambda y)]$$

where $M = \sqrt{\lambda^2 + (1/4)}$. Then the exact solution is

$$u(x,y) = \exp\left(-x\sqrt{\lambda^2 + (1/4)}\right) cos(y/2) + \exp(-\lambda y) - \exp(-\lambda\pi)$$

Here the grid is taken as $x_k = \pi k/K + 1; k = 1(1)K$ and $y_l = \pi l/K + 1; l = 1(1)K$ and $K > \lambda$. The sup norms of the error function for various λ's are tabulated in table 3.2.

Table 3.2: Sup norm of the error function in Ω of Problem2

λ	No. of Boundary Elements			
	4	8	16	32
2	1.464029	.983948	.664352	.329874
10	.963045	.529384	.279384	.076261
20	$.746531\times10^{-1}$	$.719387\times10^{-1}$	$.68390\times10^{-1}$	$.654158\times10^{-1}$
100	$.453092\times10^{-3}$	$.418373\times10^{-3}$	$.384721\times10^{-3}$	$.345127\times10^{-3}$
200	$.246721\times10^{-8}$	$.228921\times10^{-8}$	$.226018\times10^{-8}$	$.222973\times10^{-8}$
1000	$.843278\times10^{-27}$	$.837654\times10^{-27}$	$.834396\times10^{-27}$	$.831860\times10^{-27}$
2000	$.462931\times10^{-46}$	$.462745\times10^{-46}$	$.426691\times10^{-46}$	$.426672\times10^{-46}$

Example 3

$$\begin{aligned} \varepsilon^2\Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u &= f\ , \quad 0 < x, y < 1 \quad \& \quad 0 < \varepsilon \ll 1 \\ u &= \exp(-x/\varepsilon) + \exp(-y/\varepsilon) \quad \text{on the boundary.} \end{aligned}$$

Here f is given so as the problem has obvious exact solution. The numerical results are shown in table 3.3.

Table 3.3: Sup norm of the error function in Ω of Problem 3

ε	No. of Boundary Elements			
	4	8	16	32
10^{-1}	2.239343	1.694729	1.256905	1.113904
10^{-2}	$.2418854\times10^{-1}$	$.2418621\times10^{-1}$	$.2418573\times10^{-1}$	$.2417994\times10^{-1}$
10^{-3}	$.4171974\times10^{-20}$	$.4171974\times10^{-20}$	$.4171974\times10^{-20}$	$.4171974\times10^{-20}$
10^{-4}	$.311994\times10^{-206}$	$.311994\times10^{-206}$	$.311994\times10^{-206}$	$.311994\times10^{-206}$

3.7 Conclusion

In this chapter we have presented a boundary element method for general linear singular perturbation problems of elliptic type. This we have done by constructing the fundamental solution corresponding to the given problem and have given an algorithm using the dual reciprocity method to evaluate the approximate fundamental solution through only boundary integrals. The numerical results show that the method has been successfully applied to various linear singular perturbation problems and the results are very encouraging.

Chapter 4

Iterative method for non-linear Singular Perturbation Problems

4.1 Introduction

In this chapter, an iterative scheme is proposed for solving non- linear singular perturbation problem by linearizing the given problem. The linearized problem will then be solved by the approach discussed in the third chapter. The convergence of the iteration has also been established. The problem that we consider here will be of the form

$$Lu \equiv \Delta u = \lambda^2 f(x,y,u) \quad \text{in} \quad \Omega; \qquad 1 \ll \lambda < \infty \tag{4.1}$$

$$u = \bar{u} \quad \text{in} \quad \Gamma \; (= \partial\Omega) \tag{4.2}$$

We assume the conditions for the existence and uniqueness of the solution. That is, $f(x,y,u)$ is a smooth function and there exists a constant $f_1 > 0$ such that $f_u(x,y,u) \geq f_1$.[53, 5, 33]

4.2 Iterative method

The quasilinearization can be done for the given equation (4.1) and the linearized problems can then be solved using the approach discussed in the earlier chapters. But the

fundamental solution of the operator due to the quasilinearization happens to be dependent on the solution u_{m-1} at the m^{th} iteration, where u_{m-1} is the solution at the $(m-1)^{th}$ iteration. That is, at each iteration we have a different fundamental solution. If we can have a common fundamental solution for all the iterations then that will save lot of computations. This motivates us to look for a scheme which leads us to have a fundamental solution independent of u_{m-1} at the m^{th} iteration. We propose the following linearization, and in the we prove that this iteration is convergent for large λ.

$$L_1 u_m \equiv \Delta u_m - \lambda^2 u_m = \lambda^2 [f(x, y, u_{m-1}) - u_{m-1} f_u(x, y, u_{m-1})] \quad \text{in} \quad \Omega \tag{4.3}$$

$$u_m = \bar{u} \quad \text{on} \quad \Gamma \tag{4.4}$$

where we give $u_0(z)$ as the initial guess solution and using this we iteratively solve for $u_1(z)$, $u_2(z)$, $u_3(z)$ etc. In all the iterations, the left hand side operator is $[\Delta - \lambda^2]$, and we know that the fundamental solution of the operator $[\Delta - \lambda^2]$ is

$$P_0 = \frac{1}{2\sqrt{2\pi}} \left[\frac{\exp(-\lambda|z-\xi|)}{\sqrt{\lambda|z-\xi|}} \right] \tag{4.5}$$

Using this fundamental solution in the boundary element method, as discussed in chapter 3, we can solve (4.3)-(4.4). The formulation for this goes as follows.

$$\int_\Omega \Delta u_m P_0 \, d\Omega - \int_\Omega \lambda^2 u_m P_0 \, d\Omega = \int_\Omega \lambda^2 g_{m-1} p_0 \, d\Omega \tag{4.6}$$

where $g = [f(x, y, u) - u f_u(x, y, u)]$ and $g_{m-1} = [f(x, y, u_{m-1}) - u_{m-1} f_u(x, y, u_{m-1})]$.

We know from Green's identity,

$$\int_\Omega \Delta u_m \, P_0 \, d\Omega = \int_\Omega \Delta P_0 \, u_m \, d\Omega + \int_\Gamma P_0 \frac{\partial u_m}{\partial n} \, d\Gamma - \int_\Gamma u_m \frac{\partial P_0}{\partial n} \, d\Gamma$$

Using this in (4.6), we get

$$\int_\Omega \Delta P_0 \, u_m \, d\Omega + \int_\Gamma P_0 \frac{\partial u_m}{\partial n} \, d\Gamma - \int_\Gamma u_m \frac{\partial P_0}{\partial n} \, d\Gamma - \int_\Omega \lambda^2 u_m P_0 \, d\Omega = \int_\Omega \lambda^2 g_{m-1} P_0 \, d\Omega \tag{4.7}$$

$$\int_\Omega \left[\Delta P_0 - \lambda^2 P_0\right] u_m \, d\Omega + \int_\Gamma P_0 \frac{\partial u_m}{\partial n} \, d\Gamma - \int_\Gamma u_m \frac{\partial P_0}{\partial n} \, d\Gamma - \int_\Omega \lambda^2 g_{m-1} P_0 \, d\Omega = 0 \tag{4.8}$$

As P_0 is the fundamental solution of the operator $[\Delta - \lambda^2]$, we have $[\Delta - \lambda^2]P_0 = -\delta_z$, where δ_z is the Dirac delta function. Then, (4.8) becomes,

$$-u_m(z) - \int_\Gamma u_m \frac{\partial P_0}{\partial n} \, d\Gamma - \int_\Omega \lambda^2 g_{m-1} P_0 \, d\Omega = -\int_\Gamma P_0 \frac{\partial u_m}{\partial n} \, d\Gamma \tag{4.9}$$

The equation (4.9) is valid for any internal point z of Ω and it can be seen that the same expression is also valid for any boundary point. Hence for each boundary node 'i', we have

$$\int_\Gamma P_0 \frac{\partial u_m}{\partial n} \, d\Gamma = u_m^i + \int_\Gamma u_m \frac{\partial P_0}{\partial n} \, d\Gamma + \int_\Omega \lambda^2 g_{m-1} P_0 \, d\Omega \tag{4.10}$$

We use the dual reciprocity method[54] in order to transform the domain integral into boundary integral in the above equation. For this we first propose an approximation for g_{m-1} as follows.

$$g_{m-1} \simeq \sum_{j=1}^{N+L} \phi_j^{(m-1)} f_j \tag{4.11}$$

where ϕ_j's are a set of initially unknown coefficients and the f_j are approximation functions. N is the number of boundary nodes and L is the number of interior nodes. The above approximation is such that g_{m-1} is exact at these $(N + L)$ nodes. The expansion may then be considered as valid over the whole problem domain. Now, we choose a series of functions $\hat{u}_j$ such that

$$[\Delta - \lambda^2]\tilde{u}_j = f_j \tag{4.12}$$

Substituting (4.11) in (4.10), we get

$$\begin{aligned}
\int_\Gamma P_0 \frac{\partial u_m}{\partial n}\, d\Gamma &= u_m^i + \int_\Gamma u_m \frac{\partial P_0}{\partial n}\, d\Gamma + \int_\Omega \lambda^2 P_0 \left[\sum_{j=1}^{N+L} \phi_j^{(m-1)} f_j\right] d\Omega \\
&= u_m^i + \int_\Gamma u_m \frac{\partial P_0}{\partial n}\, d\Gamma + \int_\Omega \lambda^2 P_0 \left[\sum_{j=1}^{N+L} \phi_j^{(m-1)} [\Delta - \lambda^2]\tilde{u}_j\right] d\Omega \\
&= u_m^i + \int_\Gamma u_m \frac{\partial P_0}{\partial n}\, d\Gamma + \sum_{j=1}^{N+L} \phi_j^{(m-1)} \int_\Omega \lambda^2 P_0 [\Delta - \lambda^2]\tilde{u}_j\, d\Omega \qquad (4.13)
\end{aligned}$$

We can again use the Green's identity and the property of the fundamental solution P_0, to transform above domain integrals into boundary integral. Therefore, after discretization (4.13) becomes

$$\begin{aligned}
\sum_j G_{ij} \left(\frac{\partial u_m}{\partial n}\right)_j &= u_m^i + \sum_j \hat{H}_{ij}(u_m)_j \\
&\quad + \sum_{j=1}^{N+L} \phi_j^{(m-1)} \left[\tilde{u}_{ij} + \sum_{k=1}^{N} \int_{\Gamma_k} \frac{\partial P_0}{\partial n}\tilde{u}_j d\Gamma - \sum_{k=1}^{N} \int_{\Gamma_k} P_0\, \tilde{q}_j d\Gamma\right] \\
&= u_m^i + \sum_j \hat{H}_{ij}(u_m)_j \\
&\quad + \sum_{j=1}^{N+L} \phi_j^{(m-1)} \left[\tilde{u}_{ij} + \sum_{k=1}^{N} \hat{H}_{ik}\tilde{u}_{kj} - \sum_{k=1}^{N} G_{ik}\tilde{q}_{kj}\right] \\
&= u_m^i + \sum_j \hat{H}_{ij}(u_m)_j + \sum_{j=1}^{N+L} \phi_j^{(m-1)} d_{ij} \qquad (4.14)
\end{aligned}$$

where

$$\hat{H}_{ij} = \int_{\Gamma_j} \frac{\partial P_0}{\partial n}\, d\Gamma, \quad G_{ij} = \int_{\Gamma_j} P_0\, d\Gamma, \quad \tilde{q}_j = \frac{\partial \tilde{u}_j}{\partial n}, \quad d_{ij} = \left[\tilde{u}_{ij} + \sum_{k=1}^{N} \hat{H}_{ik}\tilde{u}_{kj} - \sum_{k=1}^{N} G_{ik}\tilde{q}_{kj}\right]$$

$$(q_m)_j = (\partial u_m/\partial n)_j, \quad \text{and} \quad H_{ij} = \hat{H}_{ij} \quad \text{for} \quad i \neq j$$
$$= 1 + \hat{H}_{ij} \quad \text{for} \quad i = j$$

Then (4.14) becomes

$$\sum_j G_{ij}(q_m)_j = \sum_j H_{ij}(u_m)_j + \sum_{j=1}^{N+L} \phi_j^{(m-1)} d_{ij} \tag{4.15}$$

Note that d_{ij} is a known quantity. We shall explain, in a little while, how we find $\phi^{(m-1)}$. Since we know u_m on Γ, the right hand side of (4.15) is completely known. Then writing this equation for all the boundary nodes, we get a system in matrix-vector form.

$$\boldsymbol{G}\boldsymbol{Q}_m = \boldsymbol{b}_m \tag{4.16}$$

Solving this system, we can find all the $(q_m)_j$'s. We have seen that equation (4.10) is valid for both internal and boundary node. Hence equations (4.13) and (4.14) are also valid for any internal node 'i'. Writing the equation (4.14) for an internal node 'i',

$$u_m^i = \sum_j G_{ij}(q_m)_j - \sum_j \hat{H}_{ij}(u_m)_j - \sum_{j=1}^{N+L} \phi_j^{(m-1)} [H\tilde{u}_j - G\tilde{q}_j] \tag{4.17}$$

The right hand side of (4.17) is completely known. Thus we can find the solution at any internal node.

To find the vector $\boldsymbol{\phi}^{(m-1)}$, consider (4.11). Writing this in all the $N + L$ nodes, we get a system in matrix-vector form

$$\boldsymbol{g}_{m-1} = \boldsymbol{F}\boldsymbol{\phi}^{(m-1)} \tag{4.18}$$

where each column of F consists of a vector f_j containing the values of function f_j at $(N + L)$ collocation points. As g_{m-1} is a known function, from (3.22) we have

$$\phi^{(m-1)} = F^{-1} g_{m-1} \tag{4.19}$$

Thus $\phi^{(m-1)}$ can be found. This procedure involves discretization of the boundary only. The number and the locations of the internal nodes may be defined as desired; this is generally done at the points where it is desirable to know the solutions.

4.3 Convergence

To see the convergence of the iterative method, we consider the equation (4.9). That is, for $z \in \Omega$, we have

$$u_m(z) = -\int_\Omega \lambda^2 g_{m-1} P_0 \, d\Omega + \int_\Gamma P_0 \frac{\partial u_m}{\partial n} \, d\Gamma - \int_\Gamma \frac{\partial P_0}{\partial n} u_m \, d\Gamma \tag{4.20}$$

For exact solution $u(z)$ we have from equation (4.1),

$$\Delta u - \lambda^2 u = \lambda^2 (f - u)$$

And integrating the above equation against the fundamental solution P_0,

$$\int_\Omega [\Delta u - \lambda^2 u] d\Omega = \int_\Omega \lambda^2 (f - u) P_0 d\Omega$$

From (4.6), (4.7), (4.8) and (4.9) we get

$$u(z) = -\int_\Omega \lambda^2 (f - u) P_0 \, d\Omega + \int_\Gamma P_0 \frac{\partial u}{\partial n} \, d\Gamma - \int_\Gamma \frac{\partial P_0}{\partial n} u \, d\Gamma$$

Subtracting the exact solution u from u_m, we get

$$u_m - u = -\int_\Omega \lambda^2 P_0 [g_{m-1} - (f - u)] \, d\Omega + \int_\Gamma P_0 \left[\frac{\partial u_m}{\partial n} - \frac{\partial u}{\partial n} \right] \, d\Gamma - \int_\Gamma \frac{\partial P_0}{\partial n} [u_m - u] \, d\Gamma \tag{4.21}$$

Since on Γ, $u \equiv u_m$, the third integral on the right hand side of (4.14) goes to zero. This leads (4.14) to

$$u_m - u = -\int_\Omega \lambda^2 P_0[g_{m-1} - (f-u)]\, d\Omega + \int_\Gamma P\left[\frac{\partial u_m}{\partial n} - \frac{\partial u}{\partial n}\right]\, d\Gamma$$

$$u_m - u = -\int_\Omega \lambda^2 P_0[f_{m-1} - u_{m-1}(f_u)_{m-1} - f + u)]\, d\Omega + \int_\Gamma P\left[\frac{\partial u_m}{\partial n} - \frac{\partial u}{\partial n}\right]\, d\Gamma$$

$$u_m - u = -\int_\Omega \lambda^2 P_0[f_{m-1} - f - u_{m-1}(f_u)_{m-1} + uf_u - uf_u + u)]\, d\Omega + \int_\Gamma P\left[\frac{\partial u_m}{\partial n} - \frac{\partial u}{\partial n}\right]\, d\Gamma$$

The Mean Value Theorem gives

$f_{m-1} - f = f_u(\tilde{u})\,(u_{m-1} - u)$ and $[u_{m-1} f_u(u_{m-1}) - u f_u(u)] = [\tilde{u} f_{uu}(\tilde{u}) + f_u](\tilde{u})(u_{m-1} - u)$

where $\tilde{u} = t u_{m-1} + (1-t)u, \quad 0 < t < 1$

Choose M_m such that

$$|f_u(\tilde{u})| \leq M_m$$

and

$$|\tilde{u} f_{uu}(\tilde{u}) + f_u(\tilde{u})| \leq M_m$$

Then we get,

$$|f_{m-1} - f - u_{m-1} f_u(u_{m-1}) + u f_u(u)| = 2M_m\,|u_{m-1} - u|$$

Also, let L_m be a constant such that

$$|u f_u(u) - u| < L_m$$

So, (4.21) becomes,

$$|u_m - u| < \int_\Omega \lambda^2 P_0(z-\xi, \xi, \lambda)[2M_m|u_{m-1} - u| + L_m]\, d\Omega + \int_\Gamma P_0\left|\frac{\partial u_m}{\partial n} - \frac{\partial u}{\partial n}\right|\, d\Gamma \quad (4.22)$$

Consider the second integral on the right hand side of (4.22). Since it is over the boundary Γ, we have $|z-\xi|>0$. Hence for very large λ, $P_0(z-\xi,\xi,\lambda)$ will be very small. That implies that the integral value becomes very very small. i.e.,

$$\int_\Gamma P_0(z-\xi,\xi,\lambda)\left|\frac{\partial u_m}{\partial n}-\frac{\partial u}{\partial n}\right| d\Gamma < s', \quad \text{for some small } s' \tag{4.23}$$

Now let us consider the first integral term.

$$\begin{aligned}\int_\Omega \lambda^2[2M_m|u_{m-1}-u|+L_m]P_0\, d\Omega_\xi &= \int_{\Omega\backslash C_\rho} \lambda^2[2M_m|u_{m-1}-u|+L_m]P_0\, d\Omega_\xi \\ &+ \int_{C_\rho} \lambda^2[2M_m|u_{m-1}-u|+L_m]P_0\, d\Omega_\xi \end{aligned} \tag{4.24}$$

where C_ρ is a circle of radius ρ with centre z for some small $\rho>0$. Again, in the first integral on the right hand side of (4.24), $|z-\xi|>0$ which means that value of this integral is very small for λ very large. We have therefore,

$$\int_{\Omega\backslash C_\rho} \lambda^2 P_0[2M_m\,|u_{m-1}-u|+L_m]\, d\Omega_\xi < s'' \qquad \text{for some small } s'' \tag{4.25}$$

For integral over the circle C_ρ,

$$\int_{C_\rho} \lambda^2 P_0[2M_m|u_{m-1}-u|+L_m]\, d\Omega_\xi = \lambda^2[2M_m\,|u_{m-1}(z)-u(z)|+L_m]\int_{C_\rho} P_0(z-\xi,\xi;\lambda)d\xi \tag{4.26}$$

Furthermore,

$$\begin{aligned}\int_{C_\rho} P_0(z-\xi,\xi,\lambda)d\xi &= \int_{C_\rho} \frac{1}{2\sqrt{2\pi}}\left[\frac{\exp(-\lambda|z-\xi|)}{\sqrt{\lambda|z-\xi|}}\right]d\xi \\ &= \frac{1}{2\sqrt{2\pi}}\int_0^{2\pi} d\theta \int_0^\rho \left[\frac{\exp(-\lambda t)}{\sqrt{\lambda t}}\right] t\, dt\end{aligned}$$

where we have taken the co-ordinate transformation $(\xi_1 - x) = t\cos\theta$, $(\xi_2 - y) = t\sin\theta$.

$$\begin{aligned} \int_{C_\rho} P_0(z-\xi,\xi,\lambda)d\xi &= \frac{2\pi}{2\sqrt{2\pi}}\int_0^\rho \left[\frac{\exp(-\lambda t)}{\sqrt{\lambda t}}\right] t\, dt \\ &= \frac{\sqrt{2\pi}}{2\sqrt{\lambda}}\int_0^\rho \exp(-\lambda t)\sqrt{t}\, dt \end{aligned}$$

By mean value theorem for integrals,

$$\int_{C_\rho} P_0(z-\xi,\xi,\lambda)d\xi = \sqrt{\frac{\pi}{2}}\lambda^{-1/2}\rho\exp(-\lambda\rho')\sqrt{\rho'} \qquad \text{where } 0<\rho'<\rho.$$

Substituting this value in (4.24), we get

$$\int_{C_\rho} \lambda^2 P_0\left[2M_m|u_{m-1}u| + L_m\right]\, d\Omega_\xi = M_m\lambda^{3/2}\rho\exp(\lambda\rho')\sqrt{\rho'}\,|u_{m-1}(z)-u| + \lambda^{3/2}\rho\exp(\lambda\rho')\sqrt{\rho'}L_m \tag{4.27}$$

For given λ, ρ can be chosen such that,

$$\left|2M_m\lambda^{3/2}\rho\exp(\lambda\rho')\sqrt{\rho'}\right| < r < 1 \quad \& \quad \left|L_m\lambda^{3/2}\rho\exp(\lambda\rho')\sqrt{\rho'}\right| < s''' \tag{4.28}$$

for some s''' very small. Then (4.26) becomes,

$$\int_{C_\rho} \lambda^2 P_0\left[2M_m|u_{m-1}-u| + L_m\right]\, d\Omega_\xi < r\,|u_{m-1}(z)-u| + s''' \tag{4.29}$$

This leads (4.24) to

$$\int_\Omega M_m\lambda^2|u_{m-1}-u|P_0\, d\Omega_\xi < s'' + s''' + r\,|u_{m-1}-u| \tag{4.30}$$

Now, from (4.22), (4.23) and (4.30) we have

$$\begin{aligned} |u_m - u| &< s' + s'' + s''' + r\,|u_{m-1}-u| \\ |u_m - u| &< s + r\,|u_{m-1}-u| \quad \text{where } s = s'+s''+s''' \end{aligned} \tag{4.31}$$

Then by induction on m, we have

$$|u_m - u| < r^m |u_0 - u| + (1 + r + r^2 + \ldots + r^{m-1})s \tag{4.32}$$

which ensures the convergence of $\{u_m(z)\}$.

4.4 Test Problem

As an example we solve the following problem [6].

$$\begin{aligned} \Delta u &= \lambda^2[u - g^2u^3] \quad \text{in} \quad \Omega: \quad 0 < x, y < \pi; \qquad 1 \ll \lambda < \infty. \\ u &= 0 \quad \text{in} \quad \partial\Omega \end{aligned}$$

The function $g(x, y)$ is determined so as the exact solution is

$$u(x, y) = x[\exp(-\lambda x) - \exp(-\lambda\pi]sin(y)$$

Here the reduced problem, that is the problem obtained by setting $1/\lambda = 0$, has the solution $u = 1/g$. The linearized problems are

$$\begin{aligned} \Delta u_m - \lambda^2 u_m &= \lambda^2 \left[2g^2 u_{m-1}^3\right] \quad \text{in} \quad \Omega; 1 \ll \lambda < \infty \\ u_m &= 0 \quad \text{in} \quad \partial\Omega \end{aligned}$$

We can solve the above problem and find the solution at any internal point. In this example we solve the problem by using $4, 8, 16$ boundary elements. In the domain Ω to

Table 4.1: Sup norm of the error function in Ω

λ	No. of Boundary Elements 4		8		16	
10	.965	(21)	.839	(19)	.673	(18)
20	.450	(19)	.293	(18)	.209	(18)
100	$.964\times10^{-1}$	(15)	$.737\times10^{-1}$	(14)	$.605\times10^{-1}$	(14)
200	$.405\times10^{-1}$	(15)	$.318\times10^{-1}$	(15)	$.247\times10^{-1}$	(14)
1000	$.647\times10^{-3}$	(9)	$.647\times10^{-3}$	(9)	$.647\times10^{-3}$	(19)
2000	$.483\times10^{-7}$	(6)	$.483\times10^{-7}$	(6)	$.483\times10^{-7}$	(6)

the fact that we have actually constructed the fundamental solution for general linear elliptic problems in chapter 3.

Chapter 5

Application of the method in Semiconductor Physics

5.1 Introduction

In this chapter, we propose to apply our technique to a problem in semi-conductor physics. In earlier chapters, we have seen how our method can be applied to both linear and non-linear problems. Here, we solve a non-linear system of partial differential equations using the iterative technique proposed in chapter 4.

The fundamental semiconductor device equations form a system of three second order elliptic differential equations subject to mixed Neumann-Dirichlet boundary conditions. The system consists of Poisson's equation and the continuity equations and describes potential and carrier distribution in an arbitrary semiconductor device.

Let ψ denotes the electrostatic potential, n denotes the electron density and p denotes the hole density. Then the basic semiconductor device equations are[48]

$$\varepsilon_s \Delta \psi = q(n - p - C(x,y)) \tag{5.1}$$

$$\operatorname{div}(D_n \nabla n - \mu_n n \nabla \psi) = 0 \tag{5.2}$$

$$\operatorname{div}(D_p \nabla p + \mu_p p \nabla \psi) = 0 \tag{5.3}$$

$(x, y) \in \Omega$, where Ω is a bounded domain in R^2 representing the device geometry. The equation (5.1) is the Poisson's equation and equations (5.2) and (5.3) are the electron continuity equation and hole continuity equation respectively. The other variables in the problem are

ε_s	:	semiconductor permittivity
q	:	elementary charge
D_n	:	electron diffusion coefficient
D_p	:	hole diffusion coefficient
μ_n	:	electron mobility
μ_p	:	hole mobility
$C(x,y)$	:	doping profile (i.e., the difference of the electrically active concentration of donors and the electrically active concentration of acceptors.)

The boundary $\partial\Omega$ splits up into three adjacent parts, namely $\partial\Omega_c$, $\partial\Omega_{is}$ and $\partial\Omega_{os}$ such that $\partial\Omega_c = \bigcup_{k=1}^{r} C_k$ where C_k are connected arcs and $C_i \bigcap C_j = \{\ \}$ for $i \neq j$.

Dirichlet boundary conditions for ψ, n, p are given on $\partial\Omega_c$ as follows.

$$(n - p - C(x,y))\,|_{\partial\Omega_c} = 0 \quad \text{and} \quad np|_{\partial\Omega_c} = n_i^2$$

where n_i is the intrinsic number of the semiconductor. For the potential ψ, we have

$$\psi|_{C_k} = U_T \ln\frac{n}{n_i} + U_k$$

where U_k represents the potential applied to the ohmic contacts C_k, and U_T is the thermal voltage.

5.2 Scaling of the Problem

By appropriate scaling, the above problem can be viewed as a singularly perturbed one in the following way. Assume the validity of Einstein's relations

$$\frac{D_n}{\mu_n} = \frac{D_p}{\mu_p} = U_T$$

And take the transformation called Boltzmann statistics,

$$n = n_i \exp(\psi/U_T)u, \quad p = n_i \exp(-\psi/U_T)v$$

where $u = \exp(-\phi_n/U_T)$, $v = \exp(\phi_p/U_T)$ and ϕ_n, ϕ_p are the electron and hole quasifermilevels respectively and also $u > 0$, $v > 0$. Then, (5.1) becomes,

$$\varepsilon_s \Delta\psi = q(n_i \exp(\psi/U_T)u - n_i \exp(-\psi/U_T)v - C(x,y)) \tag{5.4}$$

Again, from Boltzmann statistics, we have,

$$\begin{aligned}
D_n \nabla n - \mu_n n \nabla\psi &= U_T \mu_n \nabla(n_i \exp(\psi/U_T)u) - \mu_n n_i \exp(\psi/U_T)u\nabla\psi \\
&= U_T \mu_n n_i \nabla(\exp(\psi/U_T)u) - \mu_n n_i \exp(\psi/U_T)u\nabla\psi \\
&= U_T \mu_n n_i \left[u\ exp(\psi/U_T)\frac{\nabla\psi}{U_T} + \exp(\psi/U_T)\nabla u\right] - \mu n_i \exp(\psi/U_T)u\nabla\psi \\
&= U_T \mu_n n_i \exp(\psi/U_T)\nabla u
\end{aligned}$$

Therefore, from (5.2) we have,

$$\begin{aligned}
\operatorname{div}(D_n \nabla n - \mu_n n \nabla\psi) &= 0 \\
\operatorname{div}(U_T \mu_n n_i \exp(\psi/U_T)\nabla u) &= 0 \\
\operatorname{div}(\mu_n \exp(\psi/U_T)\nabla u) &= 0
\end{aligned} \tag{5.5}$$

Following the similar lines, we have

$$
\begin{aligned}
D_p \nabla p + \mu_p p \nabla \psi = &= U_T \mu_p \nabla (n_i \exp(-\psi/U_T) v) + \mu_p n_i \exp(-\psi/U_T) v \nabla \psi \\
&= U_T \mu_p n_i \nabla (\exp(-\psi/U_T) v) + \mu_p n_i \exp(-\psi/U_T) v \nabla \psi \\
&= U_T \mu_p n_i \left[-v \; exp(-\psi/U_T) \frac{\nabla \psi}{U_T} + \exp(-\psi/U_T) \nabla v \right] \\
&\quad + \mu_p n_i \exp(-\psi/U_T) v \nabla \psi \\
&= U_T \mu_p n_i \exp(-\psi/U_T) \nabla v
\end{aligned}
$$

And, hence from (5.3) we have,

$$
\begin{aligned}
\operatorname{div}(D_p \nabla p + \mu_p p \nabla \psi) &= 0 \\
\operatorname{div}(U_T \mu_p n_i \exp(-\psi/U_T) \nabla v) &= 0 \\
\operatorname{div}(\mu_p \exp(-\psi/U_T) \nabla v) &= 0 \qquad (5.6)
\end{aligned}
$$

Assume that C is bounded in Ω and set

$$\bar{C} = \sup_{\Omega} |C(x,y)|, \qquad D = \frac{C}{\bar{C}} \quad \text{and} \quad l = \operatorname{diam}(\Omega).$$

The dependent variables are scaled as follows.

$$\psi_s = \frac{\psi}{U_T}, \quad n_s = \frac{n}{\bar{C}}, \quad p_s = \frac{p}{\bar{C}},$$

And the independent variables are scaled as

$$x_s = \frac{x}{l}, \quad y_s = \frac{y}{l} \quad \text{for } (x_s, y_s) \in \bar{\Omega}$$

Then, equations (5.3)-(5.5) are transformed to

$$
\begin{aligned}
\varepsilon^2 \Delta \psi &= \delta^2 [\exp(\psi) u - \exp(-\psi) v] - D \\
\operatorname{div}(\exp(\psi) \nabla u) &= 0 \\
\operatorname{div}(\exp(-\psi) \nabla v) &= 0
\end{aligned}
$$

or,

$$\begin{aligned}
\varepsilon^2 \Delta\psi &= \delta^2[\exp(\psi)u - \exp(-\psi)v] - D && (5.7)\\
\Delta u + \frac{\partial\psi}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial u}{\partial y} &= 0 && (5.8)\\
\Delta v - \frac{\partial\psi}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial\psi}{\partial y}\frac{\partial v}{\partial y} &= 0 && (5.9)
\end{aligned}$$

where

$$\varepsilon = \left(\frac{\lambda_D}{l}\right)^2 = \frac{\varepsilon_s U_T}{l^2 q\bar{C}}, \qquad \text{and } \delta^2 = \frac{n_i}{\bar{C}}$$

Here λ_D is the minimal Debye length of the device. The scaled boundary conditions are

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega_{is}} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_{is}} = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega_{is}} = 0 \tag{5.10}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega_{os}} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_{os}} = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega_{os}} = 0 \tag{5.11}$$

$$u|_{C_k} = \exp(-U_k/U_T), \qquad v|_{C_k} = \exp(U_k/U_T), \tag{5.12}$$

$$\psi|_{C_k} = \ln\left[\frac{D + \sqrt{D^2 + 4\delta^4}}{2\delta^2}\right]\Bigg|_{C_k} + \frac{U_k}{U_T} \tag{5.13}$$

For modern devices $\bar{C} \geq 10^{17}$ cm^3. For silicon and silicon oxide at room temperature $T = 300K$, we have $q = 10^{-19}As$, $\varepsilon_s = 10^{-12}As/Vcm$, $U_T = 0.025V$. And with realistic value $l = 5 \times 10^{-3}$ cm, we get $\varepsilon \leq 10^{-7} \ll 1$. Therefore, the problem (5.7)-(5.13)

constitutes a singularly perturbed quasilinear elliptic system of differential equations.

Normally, the parameter $\delta^2 \leq 10^{-7} \ll 1$, too. This however gets compensated by the Dirichlet boundary conditions (5.12) and (5.13) which imply that, on C_k,

$$\begin{aligned}
\delta^2(\exp(\psi)u - \exp(-\psi)v) &= \delta^2\left\{\exp\left(\ln\left[\frac{D+\sqrt{D^2+4\delta^4}}{2\delta^2}\right] + \frac{U_k}{U_T}\right)u \right. \\
&\quad \left. - \exp\left(-\ln\left[\frac{D+\sqrt{D^2+4\delta^4}}{2\delta^2}\right] - \frac{U_k}{U_T}\right)v\right\} \\
&= \delta^2\left\{\left[\frac{D+\sqrt{D^2+4\delta^4}}{2\delta^2}\right]\exp\left(\frac{U_k}{U_T}\right)\exp\left(-\frac{U_k}{U_T}\right)\right. \\
&\quad \left. - \left[\frac{D+\sqrt{D^2+4\delta^4}}{2\delta^2}\right]^{-1}\exp\left(-\frac{U_k}{U_T}\right)\exp\left(\frac{U_k}{U_T}\right)\right\} \\
&= \delta^2\left\{\left[\frac{D+\sqrt{D^2+4\delta^4}}{2\delta^2}\right] - \left[\frac{D+\sqrt{D^2+4\delta^4}}{2\delta^2}\right]^{-1}\right\} \\
&= \left[\frac{D+\sqrt{D^2+4\delta^4}}{2}\right] - \left[\frac{2\delta^4}{D+\sqrt{D^2+4\delta^4}}\right] \\
&= O(1) \quad \text{as } \delta^2 \to 0
\end{aligned}$$

Hence, we regard δ^2 as a fixed parameter and solve the problem (5.7)-(5.13) keeping ε as the perturbation parameter, i.e., letting $\varepsilon \to 0+$.

5.3 The Iterative Method

In this section, we propose an iterative technique, an extension of the one we have used in chapter 4, to solve the singularly perturbed quasilinear elliptic system (5.7)-(5.13)

Integrate the equation (5.7), with a weighting function w_1, we get

$$\int_{\Omega}[\varepsilon^2 \Delta\psi]w_1 \, d\Omega = \int_{\Omega}[\delta^2 \exp(\psi)u - \delta^2 \exp(-\psi)v - D]w_1 \, d\Omega \tag{5.14}$$

Set $\lambda = 1/\varepsilon$, then we have,

$$\int_{\Omega}[\Delta\psi]w_1 \, d\Omega = \int_{\Omega}\lambda^2[\delta^2 \exp(\psi)u - \delta^2 \exp(-\psi)v - D]w_1 \, d\Omega$$

Subtracting $\int_{\Omega}\lambda^2\psi w_1 \, d\Omega$ on both sides,

$$\int_{\Omega}[\Delta\psi - \lambda^2\psi]w_1 \, d\Omega = \int_{\Omega}\lambda^2[\delta^2 \exp(\psi)u - \delta^2 \exp(-\psi)v - D - \psi]w_1 \, d\Omega \tag{5.15}$$

From Green's identity, we have

$$\int_{\Omega}\Delta\psi \; w_1 \, d\Omega = \int_{\Omega}\Delta w_1 \; \psi \, d\Omega + \int_{\Gamma} w_1 \frac{\partial\psi}{\partial n} \, d\Gamma - \int_{\Gamma}\psi\frac{\partial w_1}{\partial n} \, d\Gamma, \quad \text{where } \Gamma = \partial\Omega \tag{5.16}$$

Using this in (5.15), we get

$$\begin{aligned}\int_{\Omega}[\Delta w_1 - \lambda^2 w_1] \; \psi \, d\Omega \; &+ \; \int_{\Gamma} w_1 \frac{\partial\psi}{\partial n} \, d\Gamma - \int_{\Gamma}\psi\frac{\partial w_1}{\partial n} \, d\Gamma \\ &= \int_{\Omega}\lambda^2[\delta^2 \exp(\psi)u - \delta^2 \exp(-\psi)v - D - \psi]w_1 \, d\Omega\end{aligned}$$

Let $z = (x, y)$ and $\xi = (\xi_1, \xi_2)$ be points on Ω. Let us choose w_1 as the fundamental solution of the operator $[\Delta - \lambda^2]$, then,

$$\begin{aligned}\int_{\Omega}[-\delta_z(\xi)] \; \psi \, d\Omega \; &+ \; \int_{\Gamma} w_1 \frac{\partial\psi}{\partial n} \, d\Gamma - \int_{\Gamma}\psi\frac{\partial w_1}{\partial n} \, d\Gamma \\ &= \int_{\Omega}\lambda^2[\delta^2 \exp(\psi)u - \delta^2 \exp(-\psi)v - D - \psi]w_1 \, d\Omega\end{aligned}$$

(where $\delta_z(\xi)$ is the Dirac delta function)

Then, by the property of Dirac delta function,

$$\begin{aligned}\psi(z) \; = \; \int_{\Gamma} w_1 \frac{\partial\psi}{\partial n} \, d\Gamma \; &- \; \int_{\Gamma}\psi\frac{\partial w_1}{\partial n} \, d\Gamma \\ &+ \; \int_{\Omega}\lambda^2[\delta^2 \exp(\psi)u - \delta^2 \exp(-\psi)v - D - \psi]w_1 \, d\Omega\end{aligned} \tag{5.17}$$

This is the representation we get for $\psi(z)$. For $u(z)$, integrate equation (5.8) with a weighting function w_2,

$$\int_\Omega (\Delta u) w_2 \, d\Omega + \int_\Omega \frac{\partial \psi}{\partial x}\frac{\partial u}{\partial x} w \, d\Omega + \int_\Omega \frac{\partial \psi}{\partial y}\frac{\partial u}{\partial y} w \, d\Omega \;=\; 0$$

From Green's identity (5.16),

$$\begin{aligned}\int_\Omega (\Delta w_2) u \, d\Omega + \int_\Gamma w_2 \frac{\partial u}{\partial n} \, d\Gamma - \int_\Gamma u \frac{\partial w_2}{\partial n} \, d\Gamma + \int_\Omega \frac{\partial \psi}{\partial x} u \frac{\partial w_2}{\partial x} \, d\Omega &- \int_\Omega \frac{\partial^2 \psi}{\partial x^2} u w_2 \, d\Omega \\ + \int_\Gamma \frac{\partial \psi}{\partial x} u w_2 \, dy - \int_\Omega \frac{\partial \psi}{\partial y} u \frac{\partial w_2}{\partial y} \, d\Omega - \int_\Omega \frac{\partial^2 \psi}{\partial y^2} u w_2 \, d\Omega &- \int_\Gamma \frac{\partial \psi}{\partial y} u w_2 \, dx = 0\end{aligned}$$

$$\begin{aligned}\int_\Omega \left[\Delta w_2 - \frac{\partial \psi}{\partial x}\frac{\partial w_2}{\partial x} - \frac{\partial \psi}{\partial y}\frac{\partial w_2}{\partial y} - \Delta \psi w_2\right] u \, d\Omega &+ \int_\Gamma w_2 \frac{\partial u}{\partial n} \, d\Gamma \\ - \int_\Gamma u \frac{\partial w_2}{\partial n} \, d\Gamma + \int_\Gamma \frac{\partial \psi}{\partial x} u w_2 \, dy &- \int_\Gamma \frac{\partial \psi}{\partial y} u w_2 \, dx = 0\end{aligned}$$

Choosing w_2 as the fundamental solution of the operator

$$\left[\Delta - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial y} - \Delta \psi\right]$$

we get,

$$u(z) \;+\; \int_\Gamma \frac{\partial w_2}{\partial n} u \, d\Gamma - \int_\Gamma \frac{\partial \psi}{\partial x} w_2 u \, dy + \int_\Gamma \frac{\partial \psi}{\partial y} w_2 u \, dx = \int_\Gamma w_2 \frac{\partial u}{\partial n} \, d\Gamma$$

or,

$$u(z) \;=\; -\int_\Gamma \frac{\partial w_2}{\partial n} u \, d\Gamma + \int_\Gamma \frac{\partial \psi}{\partial x} w_2 u \, dy - \int_\Gamma \frac{\partial \psi}{\partial y} w_2 u \, dx + \int_\Gamma w_2 \frac{\partial u}{\partial n} \, d\Gamma \qquad (5.18)$$

Similarly, we have for $v(z)$,

$$v(z) \;=\; -\int_\Gamma \frac{\partial w_3}{\partial n} v \, d\Gamma - \int_\Gamma \frac{\partial \psi}{\partial x} w_3 v \, dy + \int_\Gamma \frac{\partial \psi}{\partial y} w_3 v \, dx + \int_\Gamma w_3 \frac{\partial v}{\partial n} \, d\Gamma \qquad (5.19)$$

where w_3 is the fundamental solution of the operator

$$\left[\Delta + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial y} + \Delta\psi\right]$$

Let us assume, for the time being, that we know ψ, $(\partial\psi/\partial n)$, u, $(\partial u/\partial n)$, v,, $(\partial v/\partial n)$ on whole of Γ. And let $\psi_0(z)$ be the initial approximation for $\psi(z)$. Then from (5.18) and (5.19) we have,

$$u_1(z) = -\int_\Gamma \frac{\partial w_2^{(0)}}{\partial n} u \, d\Gamma + \int_\Gamma \frac{\partial\psi_0}{\partial x} w_2^{(0)} u \, dy - \int_\Gamma \frac{\partial\psi_0}{\partial y} w_2^{(0)} u \, dx + \int_\Gamma w_2^{(0)} \frac{\partial u}{\partial n} \, d\Gamma \quad (5.20)$$

and,

$$v_1(z) = -\int_\Gamma \frac{\partial w_3^{(0)}}{\partial n} v \, d\Gamma - \int_\Gamma \frac{\partial\psi_0}{\partial x} w_3^{(0)} v \, dy + \int_\Gamma \frac{\partial\psi_0}{\partial y} w_3^{(0)} v \, dx + \int_\Gamma w_3^{(0)} \frac{\partial v}{\partial n} \, d\Gamma \quad (5.21)$$

Then from (5.17),

$$\psi_1(z) = \int_\Gamma w_1 \frac{\partial\psi}{\partial n} \, d\Gamma - \int_\Gamma \psi \frac{\partial w_1}{\partial n} \, d\Gamma + \int_\Omega \lambda^2[\delta^2 \left(u_1 e^{\psi_0} - v_1 e^{-\psi_0}\right) - D - \psi_0] w_1 \, d\Omega \quad (5.22)$$

We thus define the iterations,

$$u_m(z) = -\int_\Gamma \frac{\partial w_2}{\partial n}^{(m-1)} u \, d\Gamma + \int_\Gamma \frac{\partial\psi_{m-1}}{\partial x} w_2^{(m-1)} u \, dy - \int_\Gamma \frac{\partial\psi_{m-1}}{\partial y} w_2^{(m-1)} u \, dx + \int_\Gamma w_2^{(m-1)} \frac{\partial u}{\partial n} \, d\Gamma \quad (5.23)$$

$$v_m(z) = -\int_\Gamma \frac{\partial w_3}{\partial n}^{(m-1)} v \, d\Gamma - \int_\Gamma \frac{\partial\psi_{m-1}}{\partial x} w_3^{(m-1)} v \, dy + \int_\Gamma \frac{\partial\psi_{m-1}}{\partial y} w_3^{(m-1)} v \, dx + \int_\Gamma w_3^{(m-1)} \frac{\partial v}{\partial n} \, d\Gamma \quad (5.24)$$

$$\psi_m(z) = \int_\Gamma w_1 \frac{\partial \psi}{\partial n}\, d\Gamma - \int_\Gamma \psi \frac{\partial w_1}{\partial n}\, d\Gamma + \int_\Omega \lambda^2[\delta^2 \left(u_m e^{\psi_{m-1}} - v_m e^{-\psi_{m-1}}\right) - D - \psi_{m-1}] w_1 \, d\Omega \tag{5.25}$$

We know from chapter-4 that the sequence $\{\psi_m(z)\}_{m=1}^{\infty}$ converges to $\psi(z)$, the solution of (5.7).

5.4 Boundary Element Discretization

To find the solution $\psi(z)$, for $z \in \Omega$, we first define N nodes on the boundary and discretize the boundary into N elements. i.e. $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$. Then equations (5.23)-(5.25) becomes,

$$\begin{aligned} u_m(z) = -\int_{\bigcup_{j=1}^{N}\Gamma_j} \frac{\partial w_2}{\partial n}^{(m-1)} u \, d\Gamma \;&+\; \int_{\bigcup_{j=1}^{N}\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_2^{(m-1)} u \, dy \\ &-\; \int_{\bigcup_{j=1}^{N}\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_2^{(m-1)} u \, dx + \int_{\bigcup_{j=1}^{N}\Gamma_j} w_2^{(m-1)} \frac{\partial u}{\partial n} \, d\Gamma \end{aligned}$$

$$\begin{aligned} v_m(z) = -\int_{\bigcup_{j=1}^{N}\Gamma_j} \frac{\partial w_3}{\partial n}^{(m-1)} v \, d\Gamma \;&-\; \int_{\bigcup_{j=1}^{N}\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_3^{(m-1)} v \, dy \\ &+\; \int_{\bigcup_{j=1}^{N}\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} \dot{w}_3^{(m-1)} v \, dx + \int_{\bigcup_{j=1}^{N}\Gamma_j} w_3^{(m-1)} \frac{\partial v}{\partial n} \, d\Gamma \end{aligned}$$

$$\begin{aligned} \psi_m(z) \;=\; \int_{\bigcup_{j=1}^{N}\Gamma_j} w_1 \frac{\partial \psi}{\partial n} \, d\Gamma \;&-\; \int_{\bigcup_{j=1}^{N}\Gamma_j} \psi \frac{\partial w_1}{\partial n} \, d\Gamma \\ &+\; \int_\Omega \lambda^2[\delta^2 \left(u_m e^{\psi_{m-1}} - v_m e^{-\psi_{m-1}}\right) - D - \psi_{m-1}] w_1 \, d\Omega \end{aligned}$$

or,

$$u_m(z) = -\sum_{j=1}^{N}\int_{\Gamma_j} \frac{\partial w_2}{\partial n}^{(m-1)} u \, d\Gamma + \sum_{j=1}^{N}\int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_2^{(m-1)} u \, dy - \sum_{j=1}^{N}\int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_2^{(m-1)} u \, dx + \sum_{j=1}^{N}\int_{\Gamma_j} w_2^{(m-1)} \frac{\partial u}{\partial n} \, d\Gamma \quad (5.26)$$

$$v_m(z) = -\sum_{j=1}^{N}\int_{\Gamma_j} \frac{\partial w_3}{\partial n}^{(m-1)} v \, d\Gamma - \sum_{j=1}^{N}\int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_3^{(m-1)} v \, dy + \sum_{j=1}^{N}\int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_3^{(m-1)} v \, dx + \sum_{j=1}^{N}\int_{\Gamma_j} w_3^{(m-1)} \frac{\partial v}{\partial n} \, d\Gamma \quad (5.27)$$

$$\psi_m(z) = \sum_{j=1}^{N}\int_{\Gamma_j} w_1 \frac{\partial \psi}{\partial n} \, d\Gamma - \sum_{j=1}^{N}\int_{\Gamma_j} \psi \frac{\partial w_1}{\partial n} \, d\Gamma + \int_{\Omega} \lambda^2[\delta^2 \left(u_m e^{\psi_{m-1}} - v_m e^{-\psi_{m-1}}\right) - D - \psi_{m-1}] w_1 \, d\Omega \quad (5.28)$$

Call

$$b_{m-1} = \int_{\Omega} \lambda^2[\delta^2 \left(u_m e^{\psi_{m-1}} - v_m e^{-\psi_{m-1}}\right) - D - \psi_{m-1}] w_1 \, d\Omega$$

which can be evaluated through boundary integrals using the dual reciprocity method as discussed in chapter-3. Now, to evaluate integrals over Γ_j, we can approximate Γ_j to be linear, quadratic or cubic elements. For instance, if we take it to be quadratic, then on each Γ_j, we have 3 nodes, say, $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3). And the functions are approximated on Γ_j as follows.

$$\psi(s) = \sum_{c=1}^{3} N_c(s)\ \psi_{c,j} \quad (5.29)$$

$$\frac{\partial \psi(s)}{\partial n} = \sum_{c=1}^{3} N_c(s) \left(\frac{\partial \psi}{\partial n}\right)_{c,j} \quad (5.30)$$

$$u(s) = \sum_{c=1}^{3} N_c(s)\ u_{c,j} \quad (5.31)$$

$$\frac{\partial u(s)}{\partial n} = \sum_{c=1}^{3} N_c(s) \left(\frac{\partial u}{\partial n}\right)_{c,j} \quad (5.32)$$

$$v(s) = \sum_{c=1}^{3} N_c(s)\ v_{c,j} \tag{5.33}$$

$$\frac{\partial v(s)}{\partial n} = \sum_{c=1}^{3} N_c(s) \left(\frac{\partial v}{\partial n}\right)_{c,j} \tag{5.34}$$

where

$$N_1(s) = \frac{-s}{2}(1-s),\quad N_2(s) = (1+s)(1-s) \text{ and } N_3(s) = \frac{s}{2}(1+s); \qquad -1 \leq s \leq 1.$$

And $\psi_{c,j}$ is the value of ψ at the node c of the element Γ_j, and $(\partial\psi/\partial n)_{c,j}$ is the value of $(\partial\psi/\partial n)$ at the node c of Γ_j and so on. The spatial co-ordinates are related by the transformation,

$$x(s) = \sum_{c=1}^{3} N_c(s) x_c \quad \text{and} \quad y(s) = \sum_{c=1}^{3} N_c(s) y_c$$

In the boundary integrals,

$$d\Gamma = \frac{d\Gamma}{ds}\, ds = \sqrt{\left(\frac{dx(s)}{ds}\right)^2 + \left(\frac{dy(s)}{ds}\right)^2}\quad ds$$

where

$$\frac{dx(s)}{ds} = \sum_{c=1}^{3} \frac{dN_c(s)}{ds} x_c = \left(s - \frac{1}{2}\right) x_1 - 2s x_2 + \left(s + \frac{1}{2}\right) x_3$$

$$\frac{dy(s)}{ds} = \sum_{c=1}^{3} \frac{dN_c(s)}{ds} y_c = \left(s - \frac{1}{2}\right) y_1 - 2s y_2 + \left(s + \frac{1}{2}\right) y_3$$

Now, substituting equations (5.29)-(5.34) in equations (5.26)-(5.28), we get

$$\begin{aligned} u_m(z) = & - \sum_{j=1}^{N} \int_{\Gamma_j} \frac{\partial w_2}{\partial n}^{(m-1)} \left(\sum_{c=1}^{3} N_c(s)\ u_{c,j}\right) d\Gamma \\ & + \sum_{j=1}^{N} \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_2^{(m-1)} \left(\sum_{c=1}^{3} N_c(s)\ u_{c,j}\right) dy \\ & - \sum_{j=1}^{N} \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_2^{(m-1)} \left(\sum_{c=1}^{3} N_c(s)\ u_{c,j}\right) dx \end{aligned}$$

$$+ \sum_{j=1}^{N} \int_{\Gamma_j} w_2^{(m-1)} \left(\sum_{c=1}^{3} N_c(s) \left(\frac{\partial u}{\partial n} \right)_{c,j} \right) d\Gamma \tag{5.35}$$

$$\begin{aligned} v_m(z) = & - \sum_{j=1}^{N} \int_{\Gamma_j} \frac{\partial w_3}{\partial n}^{(m-1)} \left(\sum_{c=1}^{3} N_c(s)\ v_{c,j} \right) d\Gamma \\ & - \sum_{j=1}^{N} \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_3^{(m-1)} \left(\sum_{c=1}^{3} N_c(s)\ v_{c,j} \right) dy \\ & + \sum_{j=1}^{N} \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_3^{(m-1)} \left(\sum_{c=1}^{3} N_c(s)\ v_{c,j} \right) dx \\ & + \sum_{j=1}^{N} \int_{\Gamma_j} w_3^{(m-1)} \left(\sum_{c=1}^{3} N_c(s) \left(\frac{\partial v}{\partial n} \right)_{c,j} \right) d\Gamma \end{aligned} \tag{5.36}$$

$$\begin{aligned} \psi_m(z) = & \sum_{j=1}^{N} \int_{\Gamma_j} w_1 \left(\sum_{c=1}^{3} N_c(s) \left(\frac{\partial \psi}{\partial n} \right)_{c,j} \right) d\Gamma \\ & - \sum_{j=1}^{N} \int_{\Gamma_j} \left(\sum_{c=1}^{3} N_c(s)\ \psi_{c,j} \right) \frac{\partial w_1}{\partial n}\ d\Gamma + b_{m-1} \end{aligned} \tag{5.37}$$

Rearranging the summation terms, we get

$$\begin{aligned} u_m(z) = & - \sum_{j=1}^{N} \sum_{c=1}^{3} u_{c,j} \int_{\Gamma_j} \frac{\partial w_2}{\partial n}^{(m-1)} N_c(s)\ d\Gamma \\ & + \sum_{j=1}^{N} \sum_{c=1}^{3} u_{c,j} \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_2^{(m-1)} N_c(s)\ dy \\ & - \sum_{j=1}^{N} \sum_{c=1}^{3} u_{c,j} \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_2^{(m-1)} N_c(s)\ dx \\ & + \sum_{j=1}^{N} \sum_{c=1}^{3} \left(\frac{\partial u}{\partial n} \right)_{c,j} \int_{\Gamma_j} w_2^{(m-1)} N_c(s)\ d\Gamma \end{aligned} \tag{5.38}$$

$$v_m(z) = - \sum_{j=1}^{N} \sum_{c=1}^{3} v_{c,j} \int_{\Gamma_j} \frac{\partial w_3}{\partial n}^{(m-1)} N_c(s)\ d\Gamma$$

$$
\begin{aligned}
&- \sum_{j=1}^{N}\sum_{c=1}^{3} v_{c,j} \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_3^{(m-1)} N_c(s)\ dy \\
&+ \sum_{j=1}^{N}\sum_{c=1}^{3} v_{c,j} \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_3^{(m-1)} N_c(s)\ dx \\
&+ \sum_{j=1}^{N}\sum_{c=1}^{3} \left(\frac{\partial v}{\partial n}\right)_{c,j} \int_{\Gamma_j} w_3^{(m-1)} N_c(s)\ d\Gamma
\end{aligned} \tag{5.39}
$$

$$
\begin{aligned}
\psi_m(z) \quad = \quad & \sum_{j=1}^{N}\sum_{c=1}^{3} \left(\frac{\partial \psi}{\partial n}\right)_{c,j} \int_{\Gamma_j} w_1 N_c(s)\ d\Gamma \\
& - \sum_{j=1}^{N}\sum_{c=1}^{3} \psi_{c,j} \int_{\Gamma_j} N_c(s)\ \frac{\partial w_1}{\partial n}\ d\Gamma + b_{m-1}
\end{aligned} \tag{5.40}
$$

Set

$$
\begin{aligned}
H_{ij,c}^{(2)} &= \left[-\int_{\Gamma_j} \frac{\partial w_2^{(m-1)}}{\partial n} N_c(s)\ d\Gamma + \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_2^{(m-1)} N_c(s)\ dy - \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_2^{(m-1)} N_c(s)\ dx \right] \\
G_{ij,c}^{(2)} &= \int_{\Gamma_j} w_2^{(m-1)} N_c(s)\ d\Gamma \\
H_{ij,c}^{(3)} &= \left[-\int_{\Gamma_j} \frac{\partial w_3^{(m-1)}}{\partial n} N_c(s)\ d\Gamma - \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial x} w_3^{(m-1)} N_c(s)\ dy + \int_{\Gamma_j} \frac{\partial \psi_{m-1}}{\partial y} w_3^{(m-1)} N_c(s)\ dx \right] \\
G_{ij,c}^{(3)} &= \int_{\Gamma_j} w_3^{(m-1)} N_c(s)\ d\Gamma \\
H_{ij,c}^{(1)} &= \int_{\Gamma_j} N_c(s)\ \frac{\partial w_1}{\partial n}\ d\Gamma \\
G_{ij,c}^{(1)} &= \int_{\Gamma_j} w_1 N_c(s)\ d\Gamma
\end{aligned}
$$

where the subscript 'i' denotes the internal node z where we find the solution. Note that the right hand sides of the above expressions are all known, hence all the $H_{ij,c}^{(1)}$, $G_{ij,c}^{(1)}$, $H_{ij,c}^{(2)}$, $G_{ij,c}^{(2)}$, $H_{ij,c}^{(3)}$, $G_{ij,c}^{(3)}$ are known entities. Now, the equations (5.38)-(5.40) become,

$$
u_m(z) = \sum_{j=1}^{N}\sum_{c=1}^{3} u_{c,j} H_{ij,c}^{(2)} + \sum_{j=1}^{N}\sum_{c=1}^{3} \left(\frac{\partial u}{\partial n}\right)_{c,j} G_{ij,c}^{(2)} \tag{5.41}
$$

$$v_m(z) = \sum_{j=1}^{N}\sum_{c=1}^{3} v_{c,j} H^{(3)}_{ij,c} + \sum_{j=1}^{N}\sum_{c=1}^{3} \left(\frac{\partial v}{\partial n}\right)_{c,j} G^{(3)}_{ij,c} \quad (5.42)$$

$$\psi_m(z) = \sum_{j=1}^{N}\sum_{c=1}^{3} \left(\frac{\partial \psi}{\partial n}\right)_{c,j} G^{(1)}_{ij,c} - \sum_{j=1}^{N}\sum_{c=1}^{3} \psi_{c,j} H^{(1)}_{ij,c} + b_{m-1} \quad (5.43)$$

On Γ_j, we have (Fig-5.1)

$$\left.\begin{array}{lll} u_{1,j} = u_{j-1}, & u_{2,j} = u_j, & u_{3,j} = u_{j+1} \\ v_{1,j} = u_{j-1}, & v_{2,j} = u_j, & v_{3,j} = v_{j+1} \\ \psi_{1,j} = u_{j-1}, & \psi_{2,j} = u_j, & \psi_{3,j} = \psi_{j+1} \end{array}\right\} \quad (5.44)$$

Similarly,

$$\left.\begin{array}{lll} \left(\frac{\partial u}{\partial n}\right)_{1,j} = \left(\frac{\partial u}{\partial n}\right)_{j-1}, & \left(\frac{\partial u}{\partial n}\right)_{2,j} = \left(\frac{\partial u}{\partial n}\right)_{j}, & \left(\frac{\partial u}{\partial n}\right)_{3,j} = \left(\frac{\partial u}{\partial n}\right)_{j+1} \\ \left(\frac{\partial v}{\partial n}\right)_{1,j} = \left(\frac{\partial v}{\partial n}\right)_{j-1}, & \left(\frac{\partial v}{\partial n}\right)_{2,j} = \left(\frac{\partial v}{\partial n}\right)_{j}, & \left(\frac{\partial v}{\partial n}\right)_{3,j} = \left(\frac{\partial v}{\partial n}\right)_{j+1} \\ \left(\frac{\partial \psi}{\partial n}\right)_{1,j} = \left(\frac{\partial \psi}{\partial n}\right)_{j-1}, & \left(\frac{\partial \psi}{\partial n}\right)_{2,j} = \left(\frac{\partial \psi}{\partial n}\right)_{j}, & \left(\frac{\partial \psi}{\partial n}\right)_{3,j} = \left(\frac{\partial \psi}{\partial n}\right)_{j+1} \end{array}\right\} \quad (5.45)$$

Thus we can find $u_m(z), v_m(z)$, and $\psi_m(z)$ for any internal node $z \in \Omega$, provided we know all the $u_j, (\partial u/\partial n)_j$, $v_j, (\partial v/\partial n)_j$, $\psi_j, (\partial\psi/\partial n)_j$. But, note that we know $(\partial\psi/\partial n)$, $(\partial u/\partial n)$ and $(\partial v/\partial n)$ only on $\partial\Omega_{is} \bigcup \partial\Omega_{os}$ and we know u, v and ψ only on $\bigcup_k C_k$. That is, we do not know $(\partial u/\partial n)_j$, $(\partial v/\partial n)_j$, and $(\partial\psi/\partial n)_j$ for some j's and ψ_j, u_j and v_j for the remaining j's. Now, to find these unknown quantities, we go

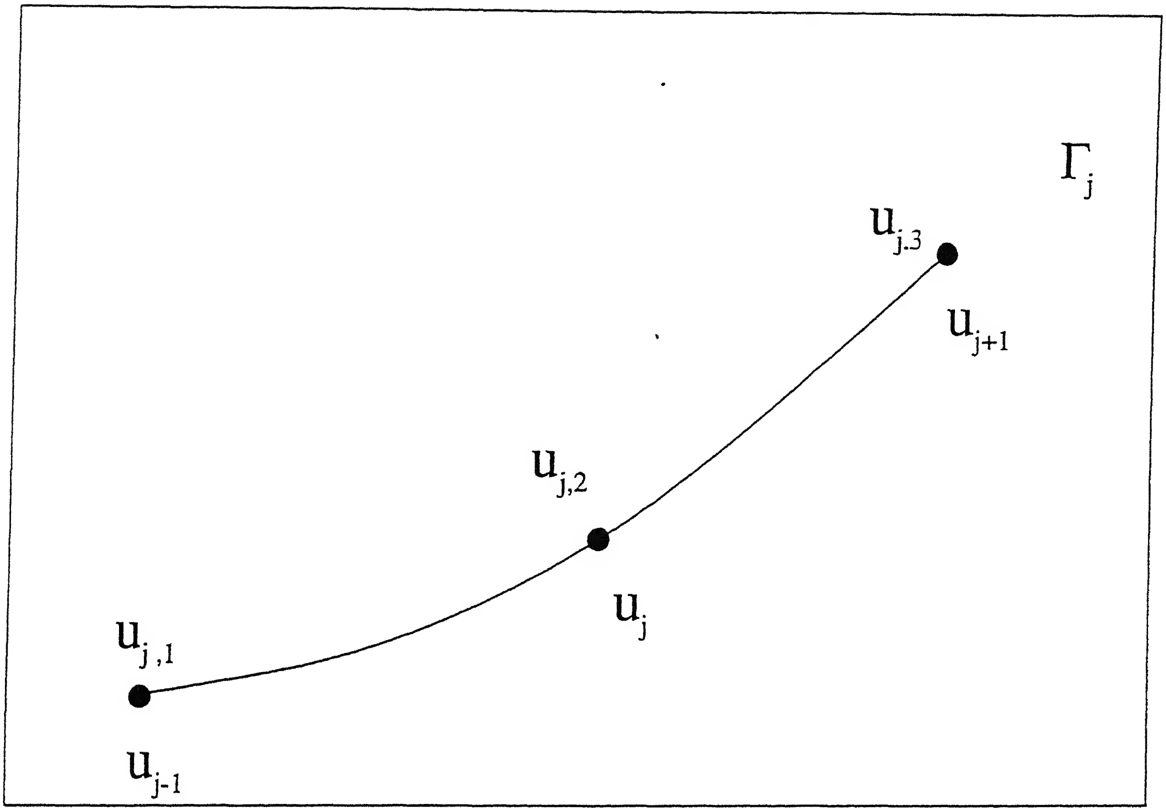


Figure 5.1: The (boundary) element Γ_j

back to the equations (5.23)-(5.25). We can see that these equations are valid for any boundary point also. Thus, equations (5.26)-(5.28) and equations (5.41)-(5.43) are also valid for any boundary node i. From (5.44) and (5.45), equation (5.41)-(5.43) can be written as

$$u_m^i = \sum_{j=1}^{N} u_j \left[H_{ij,1}^{(2)} + H_{ij,2}^{(2)} + H_{ij,3}^{(2)}\right] + \sum_{j=1}^{N} \left(\frac{\partial u}{\partial n}\right)_j \left[G_{ij,1}^{(2)} + G_{ij,2}^{(2)} + G_{ij,3}^{(2)}\right] \tag{5.46}$$

$$v_m^i = \sum_{j=1}^{N} v_j \left[H_{ij,1}^{(3)} + H_{ij,2}^{(3)} + H_{ij,3}^{(3)}\right] + \sum_{j=1}^{N} \left(\frac{\partial v}{\partial n}\right)_j \left[G_{ij,1}^{(3)} + G_{ij,2}^{(3)} + G_{ij,3}^{(3)}\right] \tag{5.47}$$

$$\psi_m^i = \sum_{j=1}^{N}\left(\frac{\partial \psi}{\partial n}\right)_j \left[G_{ij,1}^{(1)} + G_{ij,2}^{(1)} + G_{ij,3}^{(1)}\right] - \sum_{j=1}^{N}\psi_j \left[H_{ij,1}^{(1)} + H_{ij,2}^{(1)} + H_{ij,3}^{(1)}\right] + b_{m-1} \quad (5.48)$$

Set

$$\hat{H}_{ij}^{(1)} = \left[H_{ij,1}^{(1)} + H_{ij,2}^{(1)} + H_{ij,3}^{(1)}\right] \qquad G_{ij}^{(1)} = \left[G_{ij,1}^{(1)} + G_{ij,2}^{(1)} + G_{ij,3}^{(1)}\right]$$

$$\hat{H}_{ij}^{(2)} = \left[H_{ij,1}^{(2)} + H_{ij,2}^{(2)} + H_{ij,3}^{(2)}\right] \qquad G_{ij}^{(2)} = \left[G_{ij,1}^{(2)} + G_{ij,2}^{(2)} + G_{ij,3}^{(2)}\right]$$

$$\hat{H}_{ij}^{(3)} = \left[H_{ij,1}^{(3)} + H_{ij,2}^{(3)} + H_{ij,3}^{(3)}\right] \qquad G_{ij}^{(3)} = \left[G_{ij,1}^{(3)} + G_{ij,2}^{(3)} + G_{ij,3}^{(3)}\right]$$

Then writing the equations (5.46)-(5.48) for all the boundary nodes,

$$u_m^i - \sum_{j=1}^{N} u_m^j \hat{H}_{ij}^{(2)} = \sum_{j=1}^{N}\left(\frac{\partial u}{\partial n}\right)_m^j G_{ij}^{(2)}$$

$$v_m^i - \sum_{j=1}^{N} v_m^j \hat{H}_{ij}^{(3)} = \sum_{j=1}^{N}\left(\frac{\partial v}{\partial n}\right)_m^j G_{ij}^{(3)}$$

$$\psi_m^i + \sum_{j=1}^{N} \psi_m^j \hat{H}_{ij}^{(1)} = \sum_{j=1}^{N}\left(\frac{\partial \psi}{\partial n}\right)_m^j G_{ij}^{(1)} + b_{m-1}$$

Call

$$H_{ij}^{(1)} = 1 - \hat{H}_{ij}^{(1)} \qquad H_{ij}^{(2)} = 1 - \hat{H}_{ij}^{(2)} \qquad H_{ij}^{(3)} = 1 + \hat{H}_{ij}^{(3)} \qquad \text{for} \quad i = j$$

$$H_{ij}^{(1)} = \hat{H}_{ij}^{(1)} \qquad H_{ij}^{(2)} = \hat{H}_{ij}^{(2)} \qquad H_{ij}^{(3)} = \hat{H}_{ij}^{(3)} \qquad \text{for} \quad i \neq j$$

Then, we have

$$\sum_{j=1}^{N} u_m^j H_{ij}^{(2)} = \sum_{j=1}^{N}\left(\frac{\partial u}{\partial n}\right)_m^j G_{ij}^{(2)} \quad (5.49)$$

$$\sum_{j=1}^{N} v_m^j H_{ij}^{(3)} = \sum_{j=1}^{N}\left(\frac{\partial v}{\partial n}\right)_m^j G_{ij}^{(3)} \quad (5.50)$$

$$\sum_{j=1}^{N} \psi_m^j H_{ij}^{(1)} = \sum_{j=1}^{N}\left(\frac{\partial \psi}{\partial n}\right)_m^j G_{ij}^{(1)} + b_{m-1} \quad (5.51)$$

Keeping the known quantities on one side and the unknowns on the other, we get three systems of equations. which can be symbolically represented by

$$\left.\begin{aligned} A^{(1)}_m e^{(1)}_m &= h^{(1)}_m \\ \\ A^{(2)}_m e^{(1)}_m &= h^{(2)}_m \\ \\ A^{(3)}_m e^{(1)}_m &= h^{(3)}_m \end{aligned}\right\} \tag{5.52}$$

Solving these linear systems, we can find all the u^j_m, v^j_m, ψ^j_m and $(\partial u^j_m/\partial n)^j_m$, $(\partial v^j_m/\partial n)^j_m$, $(\partial \psi^j_m/\partial n)^j_m$ for all j.

5.5 Numerical Results

We have solved a two dimensional diode and presented here the numerical results. The geometry of the device is

$$\Omega = \{(x,y) : 0 < x, y < 1\}; \quad \Omega_0 = \left\{(x,y) : \sqrt{x^2+y^2} < \frac{1}{2}\right\}; \quad \Omega_1 = \Omega \backslash \Omega_0$$

$$C = \begin{cases} 10^{17}\ cm^{-3} & in\ \ \Omega_1 \\ -10^{17}\ cm^{-3} & in\ \ \Omega_0 \end{cases} \qquad l = 5 \times 10^{-3}\ cm$$

The boundary $\partial\Omega_C$ is splitted as

$$C_1 = \left\{(x,0) : 0 < x < \frac{1}{4}\right\}; \quad C_2 = \{(x,1) : 0 < x < 1\}$$

Here $\varepsilon^2 = \delta^2 = 10^{-7}$. We solve this problem[23] as discussed in the section 5.3 and 5.4. Fig-5.3 shows the potential in thermal equilibrium; i.e. $U_0 = U_1 = 0V$. Fig-5.4 shows the electron density n and Fig-5.5 shows the hole density p. These results have

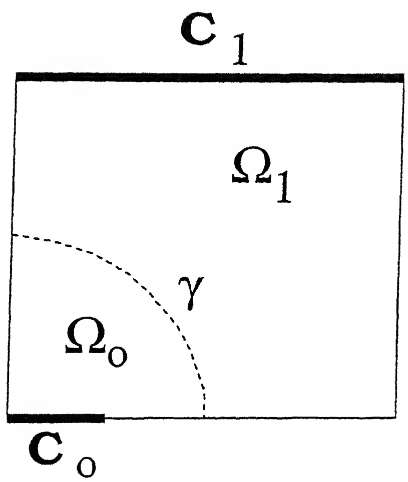


Figure 5.2: Geometry of the 2-D diode

been obtained in 20 iterations and using just 32 boundary elements.

The entities $H_{ij,c}^{(2)}$, $G_{ij,c}^{(2)}$, $H_{ij,c}^{(3)}$, $G_{ij,c}^{(3)}$, $H_{ij,c}^{(1)}$, $G_{ij,c}^{(1)}$ which are defined in terms of the boundary integrals are evaluated using 4-point Gaussian quadrature formula and resultant matrix systems (5.52) have been solved using Gauss Elimination method (with partial pivoting).

5.6 Conclusion

As it stands, boundary element method has very many advantages over domain discretization methods as it discretize only the boundary rather than domain. And, since we have proposed an iterative boundary element method which uses a fundamental solution in the asymptotic form, it efficiently solves the singularly perturbed problems. Further, the method is simple and straightforward and does not require the knowledge of the behavior of the solution expected, hence is very much suitable for those

non-mathematicians who do not appreciate the use of mathematical tools to obtain the qualitative behavior of the solution system.

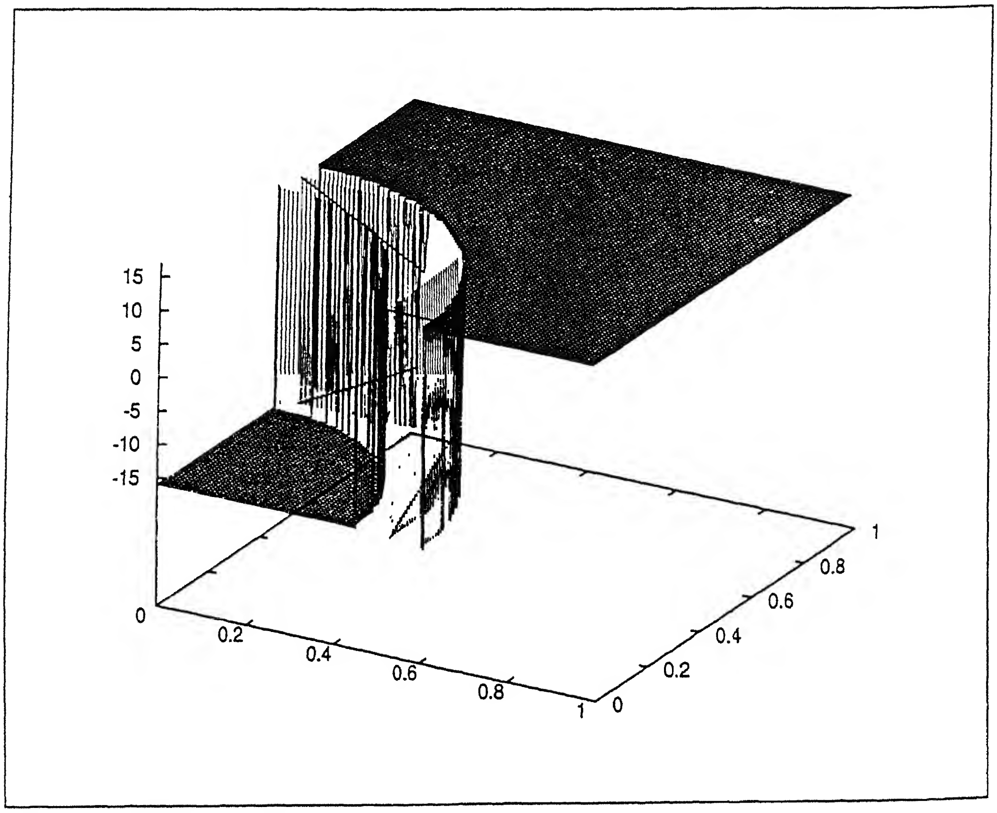


Figure 5.3: Potential diode 0 V

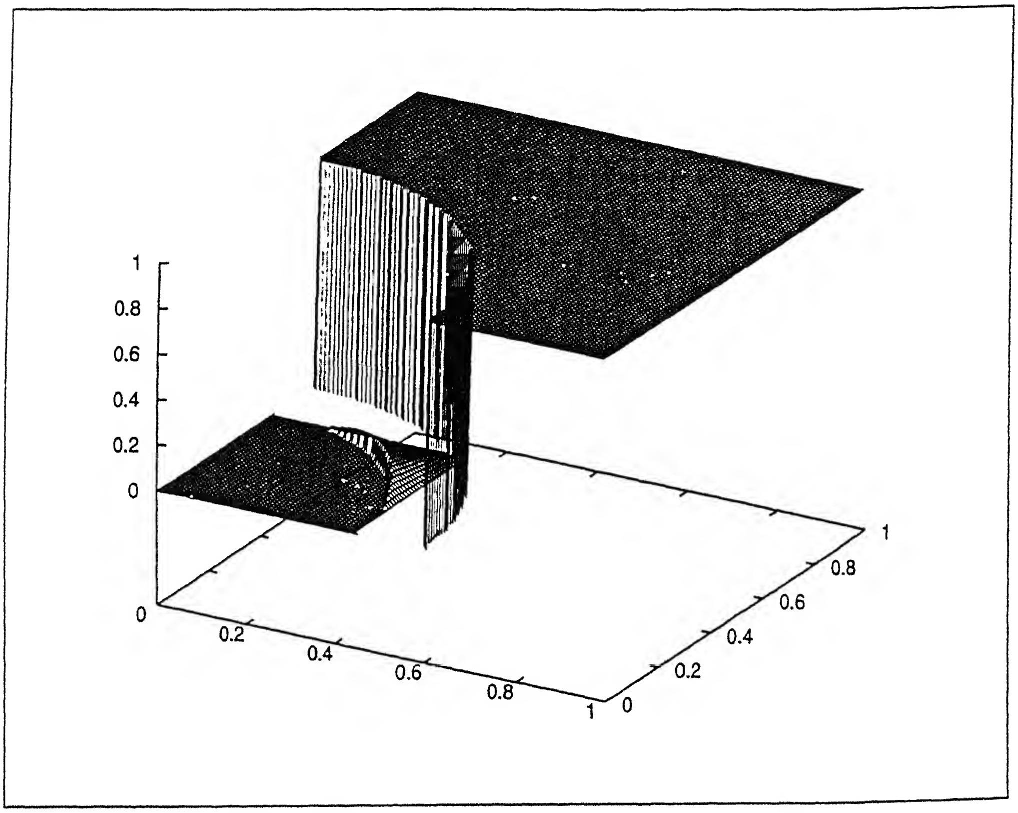


Figure 5.4: Electrons diode 0 V

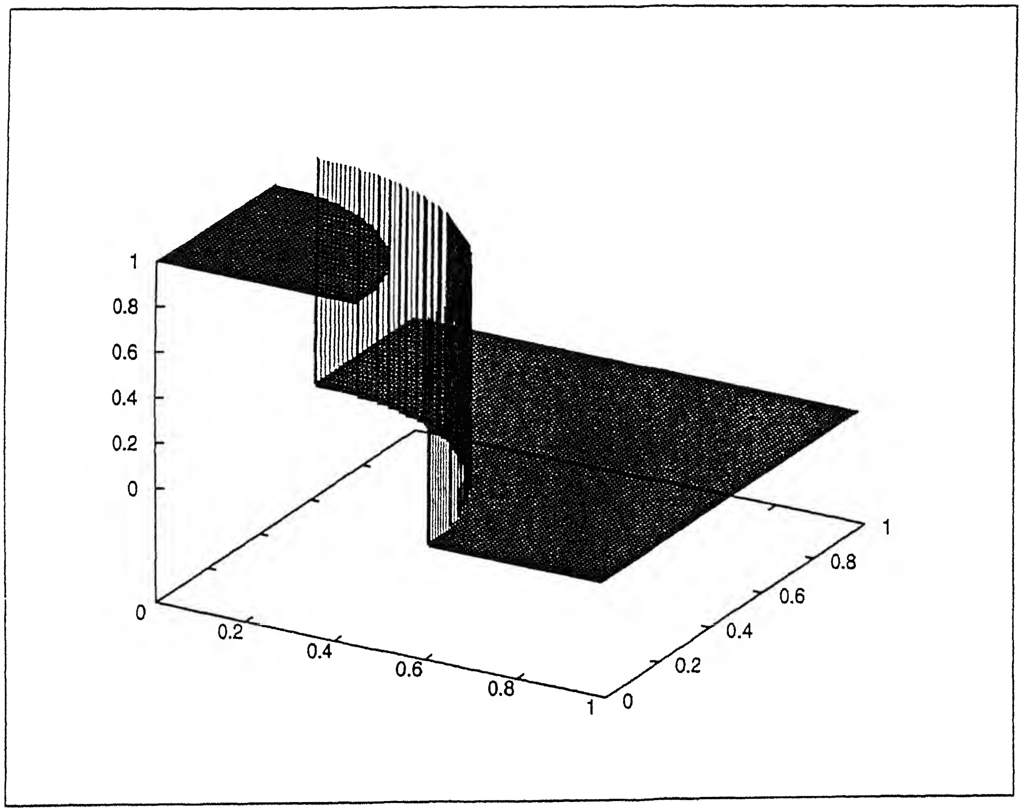


Figure 5.5: Holes diode 0 V

Bibliography

[1] W. F. AMES, *Non-linear Partial differential equations in Engineering*, Academic Press, New York, 1972.

[2] O. AXELSSON AND I. GUSTAFSSON, *Iterative solution of singular perturbation second order boundary value problems*, in Numerical Analysis of singular perturbation problems, 1979.

[3] J. BARANGER, *On the thickness of the boundary layer in elliptic singular perturbation problems*, in Numerical Analysis of singular perturbation problems, 1979.

[4] R. BELLMAN, *Perturbation Techniques in Mathematics, Physics and Engineering*. Holt, Rinehart & Winson, Newyork, 1964.

[5] R. E. BELLMAN AND R. E. KALABA, *Quasilinearization and Non-linear Boundary Value Problems*, Rand Corporation, 1965.

[6] I. P. BOGLAEV AND V. V. SIROTKIN, *The solution of singularly perturbed huge problems via domain decomposition*, Computers Math Applic., 25 (1993), pp. 31–42.

[7] A. BRANDT, *Multilevel adaptive techniques (mlat) for singular perturbation problems*, in Numerical Analysis of Singular Perturbation Problems, 1979.

[8] C. A. BREBBIA, *The boundary element method for engineers*, Pentech Press, 1978..

[9] C. A. Brebbia, J. C. F. Tells, and L. C. Wrobel, *Boundary element techniques: Theory and application in engineering*, Springer-Verlag, Berline Heidelberg Newyork Tokyo, 1984.

[10] G. F. Carrier, *Boundary Layer problems in Applied Mechanics, in Advances in Applied Mechanics III*, Academic Press, 1953.

[11] J. D. Cole, *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham Mass., 1968.

[12] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol.II Partial differential equations*, Interscience, 1962.

[13] T. A. Cruse, *Numerical solutions in three dimensional elastostatics*, Int. J. Solids and structures, (1969).

[14] ——, *Numerical evaluation of elastic stress intensity factors by the boundary integral equation method*, in The surface Crack: Physical problems and computational solutions, J. L. Swedlow (eds.), ASME, Newyork, 1972.

[15] ——, *Application of the boundary integral equation method to three dimensional stress analysis*, Computers and structures, 3 (1973), pp. 509–527.

[16] ——, *An improved boundary integral equation method for three dimensional elastic stress analysis*, Computers and structures, 4 (1974), pp. 741–754.

[17] T. A. Cruse and W. V. Buren, *Three dimensional stress analysis of a fracture specimen with an edge crack*, Int. J. Fracture Mechanics, 7 (1971), pp. 1–16.

[18] W. Eckhaus, *Matched Asymptotic Expansions and Singular Perturbations*, North-Holland, Amsterdam, 1973.

[19] ——, *Asymptotic Analysis of Singular Perturbations*, North-Holland, Amsterdam, 1979.

[20] W. Eckhaus and E. M. DeJager, *Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type*, Arch. Rational Mech. Anal., 23 (1966), pp. 26–86.

[21] T. J. R. H. (Eds.), *Finite Element Methods for Convection dominated Flows*, ASME, New York, 1979.

[22] A. Erdélyi, *Asymptotic Expansions*, Dover, New York, 1956.

[23] A. F. Franz, G. A. Franz, S. Selberherr, C. A. Ringhofer, and P. A. Morkowich, *Finite boxes: A generalization of the finite differences method suitable for some semi-conductor device simulation*, IEEE Trans on Electron Devices, ED-30 (1983), pp. 1070–1082.

[24] K. O. Friedrichs and W. Wasow, *Singular perturbations of nonlinear oscillations*, Duke Math Jl., 13 (1946), pp. 361–381.

[25] C. I. Goldstein, *Preconditioned iterative methods for convection diffusion and related boundary value problems*, Computers Math. Appln., 19 (1990), pp. 11–29.

[26] ——, *Preconditioning singularly perturbed elliptic and parabolic problems*, SIAM Jl. on Numerical Analysis, 28 (1991), pp. 1386–1418.

[27] ——, *Preconditioning convection dominated convective-diffusion problems*, Tech. Rep. BNL-49450, Rev.1/94.

[28] C. I. Goldstein and S. V. Parter, *On norm equivalence of singularly perturbed elliptic difference operators*, Appl. Math. Comp., 26 (1988), pp. 179–200.

[29] D. F. GRIFFITHS, *Towards time-stepping algorithms for convective-diffusion*, in Numerical Analysis of singular perturbation problems, Eds. P.W. Hemker and J.J.H. Miller, 1979.

[30] R. F. HARRINGTON, K. PONTOPPIDAN, P. ABRAHAMSEN, AND N. C. ALBERTSEN, *Computation of laplician potentials by an equivalent source method*, Proc. IEEE, 116 (1969), pp. 1715–1720.

[31] P. W. HEMKER AND J. J. H. M. (EDS.), *Numerical Analysis of Singular Perturbation Problems*, Academic Press, New York, 1979.

[32] J. L. HESS AND A. M. O. SMITH, *Calculation of potential theory flow about arbitrary bodies*, in Progress in Aeronautical Sciences, Vol.8, D.Kuchemann (eds.), Pergamon Press, Oxford, 1967.

[33] F. A. HOWES, *Singularly perturbed semilinear elliptic boundary value problems*, Comm. Partial Differential Equations, 4 (1979), pp. 1–39.

[34] K. INAYAT NOOR AND M. ASLAM NOOR, *Iterative methods for a class of variational inequalities*, in Numerical Analysis of singular perturbation problems, 1979.

[35] M. A. JASWON, *Integral equation methods in potential theory-i*, Proc. Roy. Soc. London, A275 (1963), pp. 23–32.

[36] M. A. JASWON AND A. R. PONTER, *An integral equation solution of the torsion problem*, Proc. Roy. Soc. London, A275 (1963), pp. 237–246.

[37] K. E. JORDON, *A numerical treatment of singularly perturbed boundary and initial value problems*, PhD thesis, Univ. of Deleware, 1980.

[38] M. K. Kadalbajoo and Y. N. Reddy, *Asymptotic and numerical analysis of singular perturbation problems: A survey*, Appl. Math. Comp., 30 (1989), pp. 223–259.

[39] S. Kaplaun, *Fluid Mechanics and Singular Perturbations (Lagerstrom, P.A.* et al.*Eds.)*, Academic Press, New York, 1967.

[40] O. D. Kellog, *Foundations of Potential theory*, Springer, Berlin, 1929.

[41] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics*, Springer-Verlag, Newyork, 1981.

[42] O. D. Kupradze, *Potential methods in the theory of elasticity*, 1965.

[43] J. C. Lachat, *A further development of the boundary integral technique for elastostatics*, PhD thesis, Univ. of Souampton, UK, 1975.

[44] J. C. Lachat and J. O. Watson, *Effective numerical treatment of boundary integral equations: a formulation for three-dimensional elastostatics*, Int. J. Numer. Meth. in Engg, 10 (1976), pp. 991–1005.

[45] N. Levinson, *The first boundary value problem for* $\varepsilon\Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)$ *for small* ε, Ann. of Math., 51 (1950), pp. 428–445.

[46] R. E. Meyer and S. V. P. (Eds.), *Singular Perturbations and Asymptotics*, Academic Press, New York, 1980.

[47] S. G. Mikhlin, *Integral Equations*, Pergamon Press, Oxford, 1957.

[48] P. A. Morkowich, C. A. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, Newyork, 1990.

[49] J. D. MURRAY, *Asymptotic analysis*, Clarendon Press, Oxford, 1974.

[50] N. I. MUSKHELISHVILI, *Some basic problems of the mathematical theory of Elasiticity*, Noordhoff, Holland, 1953.

[51] A. H. NAYFEY, *Introduction to Perturbation Techniques*, Wiley, Newyork, 1981.

[52] R. E. O'MALLEY, *Introduction to Singular Perturbations*, Academic Press, New York, 1974.

[53] J. M. ORTEGA AND W. C. RHEINBOLDT, *Iterative solutions of non-linear equations in several variables*, Academic Press, 1970.

[54] P. W. PARTRIDGE, C. A. BREBBIA, AND L. C. WROBEL, *The Dual Reciprocity boundary element method*, Comp. Mechanics Publications, London, 1992.

[55] L. PRANDTL, *über flüssigkeit-bewegung bei kleiner reibung,*, in Verh, III Int. Math. Kongresses, Tuebner Leipzig, 1905, pp. 484–491.

[56] N. RAMANUJAM AND V. M. S. KUMARI, *Method of lines for elliptic differential equations with a small parameter*, SIAM Jl. on Numerical Analysis, 25 (1988).

[57] M. L. RASULOV, *Methods of Contour Integration*, North-Holland Publishing Company, Amsterdam, 1967.

[58] F. J. RIZZO AND D. J. SHIPPY, *A formulation and solution procedure for the general non-homogeneous elastic inclusion problem*, Int. J. solids and structures, 4 (1968), pp. 1161–1179.

[59] A. H. SCHATZ AND L. B. WAHLBIN, *On the finite element method for singularly perturbed reaction-diffusion problems in two and one dimensions*, Mathematics of Computation, 40 (1983), pp. 47–89.

[60] T. SYMM, *Integral equation methods in potential theory-ii*, Proc. Roy. Soc. London, A275 (1963), pp. 33–46.

[61] C. L. TAN AND R. T. FENNER, *Elastic frature mechanics analysis of a cracked thick-walled cylinder*, J. Strain Analysis, 18 (1979), pp. 253–260.

[62] M. VAN DYKE, *Perturbation Methods in Fluid Mechanics*, Academic Press, Newyork, 1964.

[63] M. V. VELDHUIZEN, *Higher order schemes of positive type for singular perturbation problems*, in Numerical Analysis of singular perturbation problems, 1979.

[64] M. I. VISIK AND L. A. LYUSTERNIK, *Regular degenerations and boundary layer for linear differential equations with small parameter*, American Math Soc. Translations, 20 (1962), pp. 239–364.

[65] W. WASOW, *Asymptotic solution of boundary value problems for the differential equation* $\Delta u - \lambda \frac{\partial u}{\partial x} = \lambda f(x, y)$, Duke Math Jl., 11 (1944).

List of Publications

1. M. K. KADALBAJOO AND K. ARUMUGAM, Boundary element method for elliptic differential equations with a small parameter. *Applied Math. Computations*, 80:1–13, 1995.

2. K. ARUMUGAM AND M. K. KADALBAJOO, On fundamental solutions in boundary element method for singular perturbation problems. *Boundary Element Communications*, 8:14–17, 1997.

3. M. K. KADALBAJOO AND K. ARUMUGAM, Boundary element method for singularly perturbed elliptic boundary value problems. *(communicated)*.

4. K. ARUMUGAM AND M. K. KADALBAJOO, On non-linear singular perturbation problems of elliptic type using the boundary element method. *(communicated)*.

5. K. ARUMUGAM AND M. K. KADALBAJOO, Asymptotic Boundary Element Method for Semiconductor Device Equations *(communicated)*.